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SINGULARLY PERTURBED DIFFERENTIAL/ALGEBRAIC EQUATIONS

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Abstract: In this paper, singularly perturbed nonlinear differential/algebraic equations (DAEs) are considered and a proof of the existence and uniqueness of a solution is given. Asymptotic expansions for such a solution are obtained and proved to be uniformly convergent. This generalizes known results about asymptotic expansions of singularly perturbed ordinary differential equations.

AMS subject classification: 41A60, 34D15, 34E05

1. Introduction

We consider the asymptotic behavior of solutions of a singularly perturbed DAE of the form

$$\begin{aligned} x' &= f_1(x,y,z,\epsilon) \\ \epsilon y' &= f_2(x,y,z,\epsilon) \\ 0 &= f_3(x,y,z,\epsilon) \end{aligned} \tag{1.1a}$$

together with the initial conditions

$$x(0,\epsilon) = \xi(\epsilon), \quad y(0,\epsilon) = \eta(\epsilon), \quad z(0,\epsilon) = \zeta(\epsilon) \tag{1.1b}$$

where $x \in R^m$, $y \in R^n$, $z \in R^k$ and $\epsilon \in R^1$. It might be expected that the theorems developed for singularly perturbed ODEs can be used to study the singularly perturbed DAE (1.1). For this it would be natural to apply a standard index reduction to (1.1a,b); that is to reduce (1.1a,b) to a singularly perturbed ODE by differentiating the constraint equation $f_3(x(t,\epsilon), y(t,\epsilon), z(t,\epsilon), \epsilon) = 0$ with respect to t along any solution $(x(t,\epsilon),\ y(t,\epsilon),\ z(t,\epsilon))$ of (1.1a,b). Then by applying the known theory for singularly perturbed ODE's (see [Ho] or [OM2]), we might expect to obtain the desired results for the singularly perturbed DAE

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(1.1a,b). But this is not the case. In fact, by differentiating the constraint equation in the system (1.1a,b) with respect to t along the solution $(x(t,\epsilon), y(t,\epsilon), z(t,\epsilon))$, we obtain

$$
\begin{aligned}
x' &= f_1(x,y,z,\epsilon), \\
\epsilon y' &= f_2(x,y,z,\epsilon), \\
\epsilon z' &= -(D_z f_3(x,y,z,\epsilon))^{-1}(\epsilon D_x f_3(x,y,z,\epsilon) f_1(x,y,z,\epsilon), \\
&\quad + D_y f_3(x,y,z,\epsilon) f_2(x,y,z,\epsilon)), \\
x(0,\epsilon) &= \xi(\epsilon), \quad y(0,\epsilon) = \eta(\epsilon), \quad z(0,\epsilon) = \zeta(\epsilon).
\end{aligned}
\tag{1.2}
$$

For $\epsilon = 0$, the reduced problem for (1.2) then has the form:

$$
\begin{aligned}
x' &= f_1(x,y,z,0), \\
0 &= f_2(x,y,z,0), \\
x(0) &= \xi(0).
\end{aligned}
\tag{1.3}
$$

This reduced system has lost k constraint conditions which means that the assumptions of the theorems in [Ho] or [OM2] for singularly perturbed ODEs are not satisfied. Therefore, we have to study the singularly perturbed DAE (1.1) directly.

Under certain assumptions, we prove that (1.1a,b) has a unique solution on the interval $[0,T]$ for all small ϵ, for which asymptotic expansions have been obtained and proved to be convergent uniformly in $[0,T]$.

As background for the presentation, Chapter 2 below presents a summary of some known existence results for DAE's, which can be applied to (1.1) and its reduced system.

Chapter 3 addresses the limiting problems in which the reduced problem and the inner and outer problems for (1.1) are introduced, and the regular degeration of (1.1) is defined as well.

While asymptotic expansions and existence of the outer solutions are considered in Chapter 4 and 5, respectively, inner solutions of (1.1) are studied in Chapter 6 and 7.

2. Background on DAE's

To ensure the existence of solutions of (1.1), we impose the following assumption on the system

Assumption (I): *There are non-empty open sets* $\mathcal{D}_x \subset R^m$, $\mathcal{D}_y \subset R^n$, $\mathcal{D}_z \subset R^k$ and $\mathcal{J}_\epsilon \subset R^1$, $\mathcal{J}_\epsilon = \{ \epsilon \mid |\epsilon| < \epsilon',\ \epsilon' > 0\}$, *such that the mappings* $f_1 : \mathcal{D} \times \mathcal{J}_\epsilon \to R^m$, $f_2 : \mathcal{D} \times \mathcal{J}_\epsilon \to R^n$, $f_3 : \mathcal{D} \times \mathcal{J}_\epsilon \to R^k$, $\xi : \mathcal{J}_\epsilon \to \mathcal{D}_x$, $\eta : \mathcal{J}_\epsilon \to \mathcal{D}_y$ and $\zeta : \mathcal{J}_\epsilon \to \mathcal{D}_z$, where $\mathcal{D} = \mathcal{D}_x \times \mathcal{D}_y \times \mathcal{D}_z$, are

continuous on the indicated domains. Moreover for fixed $\epsilon \in \mathcal{J}_\epsilon$, f_1, f_2, f_3 are of class C^∞ on $\mathcal{D}$ and the initial point $(\xi(\epsilon),\ \eta(\epsilon),\ \zeta(\epsilon))$ satisfies the compatibility condition

$$f_3(\xi(\epsilon), \eta(\epsilon), \zeta(\epsilon), \epsilon) = 0.$$

Finally, assume that $f_2(x,y,z,0) \not\equiv 0$ and $f_3(x,y,z,0) \not\equiv 0$ in the domain $\mathcal{D}$, and that the Jacobian matrix

$$D_z f_3(\xi_0, \eta_0, \zeta_0, 0) \tag{2.1}$$

is nonsingular, where $\xi_0 = \xi(0)$, $\eta_0 = \eta(0)$, $\zeta_0 = \zeta(0)$.

The infinite differentiability of f_1, f_2, f_3 is assumed here only for the sake of simplicity. We are interested in the existence of solutions of (1.1a,b) on some interval $[0,T]$ where T is independent of ϵ, and with the asymptotic behavior of the solutions of (1.1a,b) as ϵ tends to zero. For these asymptotic considerations some further conditions will be needed which will be stated in the next subsection.

For the existence of solutions of (1.1a,b) the condition (2.1) in Assumption (I) ensures the solvability of (1.1a,b). Indeed, from (2.1) it follows that there exists an $\epsilon' > 0$ ($\epsilon' \in \mathcal{J}_\epsilon$) such that

$$D_z f_3(\xi(\epsilon), \eta(\epsilon), \zeta(\epsilon), \epsilon)$$

is nonsingular for any fixed ϵ, $0 \le \epsilon \le \epsilon'$. This implies that the system (1.1a,b) is a DAE of index one in some neighborhood of $(\xi(\epsilon), \eta(\epsilon), \zeta(\epsilon))$.

The existence and uniqueness of a solution of (1.1a,b) will be based on the following existence theorem for the solutions of initial value problem of the form:

$$\begin{aligned} u' &= F_1(u,v), \\ 0 &= F_2(u,v), \\ u(0) = u_0&, \quad v(0) = v_0, \end{aligned} \tag{2.2}$$

Proposition 2.1: *Suppose that the mappings $F_1 : \mathcal{D}_u \times \mathcal{D}_v \subset R^{r+s} \to R^r$ and $F_2 : \mathcal{D}_u \times \mathcal{D}_v \subset R^{r+s} \to R^s$ are of class C^1 on their domains where $\mathcal{D}_u \subset R^r$ and $\mathcal{D}_v \subset R^s$ are non-empty open sets, and that the initial point $(u_0, v_0) \in \mathcal{D}_u \times \mathcal{D}_v$ satisfies*

$$F_2(u_0, v_0) = 0,$$

and the Jacobian matrix

$$D_v F_2(u_0, v_0)$$

is nonsingular. Then there exists a C^1-solution $(u(t), v(t)) \in \mathcal{D}_u \times \mathcal{D}_v$ of (2.2) which is unique on some interval $[0, T_0]$, $T_0 > 0$. Moreover, the component $u(t)$ is of class C^2 on $[0, T_0]$.

Proof: The result is a consequence of the existence theorem for DAEs of the form (2.2) given in see [Rh4]. ∎

By applying Proposition 1.1 to (1.1a,b), we obtain the following existence theorem:

Proposition 2.2: *Under assumption (I), for any fixed $\epsilon > 0$ ($\epsilon \le \epsilon' \in \mathcal{J}_\epsilon$), there exists a unique solution $(x(t,\epsilon), y(t,\epsilon), z(t,\epsilon))$ for the DAE (1.1a,b) on some interval $[0, T_\epsilon]$, where T_ϵ depends on ϵ.*

3. The boundary problems

In order to study the asymptotic behavior of solutions of (1.1), we formally set $\epsilon = 0$ in (1.1a) and remove the initial conditions for y and z, and then obtain the system

$$\begin{aligned} X_0' &= f_1(X_0, Y_0, Z_0, 0), \\ 0 &= f_2(X_0, Y_0, Z_0, 0), \\ 0 &= f_3(X_0, Y_0, Z_0, 0), \end{aligned} \tag{3.1a}$$

with which we associate initial conditions of the form

$$X_0(0) = \xi_0, \quad Y_0(0) = Y_0^0, \quad Z_0(0) = Z_0^0. \tag{3.1b}$$

Thus we obtain from Proposition 1.2 the existence result:

Proposition 3.1: *Under assumption (I) let $(\xi_0, Y_0^0, Z_0^0) \in \mathcal{D}$, $\xi_0 = \xi(0)$, be a point such that*

$$\begin{aligned} f_2(\xi_0, Y_0^0, Z_0^0, 0) = 0, \\ f_3(\xi_0, Y_0^0, Z_0^0, 0) = 0 \end{aligned}$$

and that the matrices

$$D_z f_3(\xi_0, Y_0^0, Z_0^0, 0) \tag{3.2a}$$

$$\begin{aligned} B^0 =& D_y f_2(\xi_0, Y_0^0, Z_0^0, 0) \\ &- D_z f_2(\xi_0, Y_0^0, Z_0^0, 0) \left(D_z f_3(\xi_0, Y_0^0, Z_0^0, 0)\right)^{-1} D_y f_3(\xi_0, Y_0^0, Z_0^0, 0) \end{aligned} \tag{3.2b}$$

are nonsingular. Then the system (3.1) has a unique solution $(X_0(t),\ Y_0(t),\ Z_0(t)) \in \mathcal{D}$ on some interval $[0, T]$, which satisfies the initial condition (3.1b). Moreover, for $t \in [0, T]$ the matrices

$$D_z f_3(X_0(t), Y_0(t), Z_0(t), 0), \tag{3.3a}$$

$$B(t) = D_y f_2(\xi_0, Y_0(t), Z_0(t), 0) - D_z f_2(\xi_0, Y_0(t), Z_0(t), 0) \\ \left(D_z f_3(\xi_0, Y_0(t), Z_0(t), 0)\right)^{-1} D_y f_3(\xi_0, Y_0(t), Z_0(t), 0) \tag{3.3b}$$

are nonsingular.

Proof: Since the matrices (3.2a/b) are nonsingular, there exists a neighborhood $O_\delta \subset \mathcal{D}$ of the point (ξ_0, Y_0^0, Z_0^0) such that for all $(x, y, z) \in O_\delta$ the matrices

$$D_z f_3(x, y, z, 0),$$

$$D_y f_2(x, y, z, 0) - D_z f_2(x, y, z, 0)\left(D_z f_3(x, y, z, 0)\right)^{-1} D_y f_3(x, y, z, 0)$$

are nonsingular. This implies that the Jacobian matrix

$$\begin{pmatrix} D_y f_2(x, y, z, 0) & D_z f_2(x, y, z, 0) \\ D_y f_3(x, y, z, 0) & D_z f_3(x, y, z, 0) \end{pmatrix} \tag{3.4}$$

is nonsingular in O_δ, as follows directly from the identity

$$\begin{pmatrix} D_y f_2 & D_z f_2 \\ D_y f_3 & D_z f_3 \end{pmatrix}$$

$$= \begin{pmatrix} I_{m\times m} & D_z f_2 (D_z f_3)^{-1} \\ 0 & I_{k\times k} \end{pmatrix} \begin{pmatrix} D_y f_2 - D_z f_2 (D_z f_3)^{-1} D_y f_3 & 0 \\ D_y f_3 & D_z f_3 \end{pmatrix}.$$

Then a direct application of the Proposition 2.1 to the system (3.1a/b) shows that this system has a unique solution $(X_0(t), Y_0(t), Z_0(t)) \in O_\delta \in \mathcal{D}$ on some interval $[0, T]$. Since for all $t \in [0, T]$, the solution $(X_0(t), Y_0(t), Z_0(t))$ remains in O_δ, the matrices

$$D_z f_3(X_0(t), Y_0(t), Z_0(t), 0) \quad \text{and } B(t)$$

are nonsingular for all $t \in [0, T]$. ∎

Our aim will be to determine when there are solutions of (1.1) that converge for $\epsilon \to 0$ to a solution of the reduced system (3.1). For this we introduce the following concept:

Definition 3.1: *The system (1.1a,b) is said to degenerate regularly on the solution $(X_0(t)$, $Y_0(t)$, $Z_0(t))$ of (3.1), $0 \le t \le T$, if a solution $(x(t, \epsilon), y(t, \epsilon), z(t, \epsilon))$ of (1.1) exists on the same interval $0 \le t \le T$, which converges to $(X_0(t), Y_0(t), Z_0(t))$ as $\epsilon \to 0$, uniformly on compact subsets of $0 < t \le T$.*

The structure of regularly degenerating solution of (1.1a,b) is determined by replacing problem (1.1a,b) by two auxiliary problems; the first of these is called the outer problem, and the second one the inner problem.

The critical idea is here to consider (1.1a) with only an intial condition for x but with the explicit assumption that only solutions are admitted which for $\epsilon = 0$ reduce to a solution of (3.1). In other words, we consider the problem:

$$\begin{aligned} X' &= f_1(X, Y, Z, \epsilon), \\ \epsilon Y' &= f_2(X, Y, Z, \epsilon), \\ 0 &= f_3(X, Y, Z, \epsilon), \end{aligned} \tag{3.5a}$$

with some initial condition

$$X(0, \epsilon) = \xi^*(\epsilon) \tag{3.5b}$$

and the limiting assumption

$$X(t, 0) = X_0(t), \; Y(t, 0) = Y_0(t), \; Z(t, 0) = Z_0(t) \tag{3.5c}$$

where $(X_0(t), Y_0(t), Z_0(t))$ is a solution of (3.1).

Any solution of (3.5a,b,c) will be called an outer solution. With any such outer solution $(X(t,\epsilon), Y(t,\epsilon), Z(t,\epsilon))$ we introduce in (1.1a) the scaled variable

$$\tau = t/\epsilon \tag{3.6a}$$

and new dependent functions

$$\begin{aligned} \alpha(\tau, \epsilon) &= x(\epsilon\tau, \epsilon) - X(\epsilon\tau, \epsilon) \\ \beta(\tau, \epsilon) &= y(\epsilon\tau, \epsilon) - Y(\epsilon\tau, \epsilon) \\ \gamma(\tau, \epsilon) &= z(\epsilon\tau, \epsilon) - Z(\epsilon\tau, \epsilon) \end{aligned} \tag{3.6b}$$

Let $(x(t,\epsilon), y(t,\epsilon), z(t,\epsilon))$ be a solution of (1.1a,b), then we find that (α, β, γ) satisfies the following DAE, which is called the boundary layer problem or inner problem:

$$\begin{aligned} \frac{d\alpha}{d\tau} &= \epsilon \hat{f}_1(\epsilon\tau, \alpha, \beta, \gamma, \epsilon), \\ \frac{d\beta}{d\tau} &= \hat{f}_2(\epsilon\tau, \alpha, \beta, \gamma, \epsilon), \\ 0 &= \hat{f}_3(\epsilon\tau, \alpha, \beta, \gamma, \epsilon), \end{aligned} \tag{3.7a}$$

$$\alpha(0,\epsilon) = \xi(\epsilon) - \xi^*(\epsilon), \; \beta(0,\epsilon) = \eta(\epsilon) - Y(0,\epsilon), \; \gamma(0,\epsilon) = \zeta(\epsilon) - Z(0,\epsilon),$$

where

$$\begin{aligned} \hat{f}_i(t, \alpha, \beta, \gamma, \epsilon) = {} & f_i(\alpha + X(t,\epsilon), \beta + Y(t,\epsilon), \gamma + Z(t,\epsilon), \epsilon) \\ & - f_i(X(t,\epsilon), Y(t,\epsilon), Z(t,\epsilon), \epsilon), \quad i = 1, 2, 3. \end{aligned} \tag{3.7b}$$

To study the asymptotic behavior of solutions of (1.1), we need following assumption

Assumption (II): *The components of the initial point $(\xi(\epsilon), \eta(\epsilon), \zeta(\epsilon))$ possess asymptotic expansions :*

$$\xi(\epsilon) \sim \sum_{i=0}^{\infty} \xi_i \epsilon^i,\ \eta(\epsilon) \sim \sum_{i=0}^{\infty} \eta_i \epsilon^i,\ \zeta(\epsilon) \sim \sum_{i=0}^{\infty} \zeta_i \epsilon^i \qquad \text{as } \epsilon \to 0. \tag{3.8}$$

4. Asymptotic expansions of outer solutions

For the analysis of the solutions of (3.5a,b) and (3.7a,b) and their interrelationship, asymptotic considerations are to be used. We motivate here briefly the approach and defer proofs to the next subsection. Suppose that the initial function $\xi^*(\epsilon)$ of (3.5b) satisfies

$$\xi^*(\epsilon) \sim \xi_0 + \sum_{i=1}^{\infty} \xi_i^* \epsilon^i, \quad \text{as } \epsilon \to 0, \tag{4.1a}$$

and accordingly that any outer solution $(X(t,\epsilon), Y(t,\epsilon), Z(t,\epsilon))$ has a formal asymptotic expansion in terms of ϵ

$$\begin{aligned} X(t,\epsilon) &= \sum_{i=0}^{N} X_i(t)\epsilon^i + O(\epsilon^{N+1}) \quad \text{as } \epsilon \to 0 \\ Y(t,\epsilon) &= \sum_{i=0}^{N} Y_i(t)\epsilon^i + O(\epsilon^{N+1}) \quad \text{as } \epsilon \to 0 \\ Z(t,\epsilon) &= \sum_{i=0}^{N} Z_i(t)\epsilon^i + O(\epsilon^{N+1}) \quad \text{as } \epsilon \to 0, \end{aligned} \tag{4.1b}$$

which is assumed to hold uniformly for $0 \le t \le T$, $0 < \epsilon \le \epsilon_1$ $(\epsilon_1 \le \epsilon')$.

Inserting (4.1b) into the equation (3.5a), expanding the right side functions at the point $(X_0(t), Y_0(t), Z_0(t), 0)$ and equating coefficients of equal powers of ϵ, we obtain that the first term $(X_0(t), Y_0(t), Z_0(t))$ must be a solution of the reduced problem (3.1a.b) while the higher terms $(X_r(t), Y_r(t), Z_r(t))$, $r = 1, ..., N$ in (4.1b) must satisfy a linear DAE of the form

$$\begin{aligned} \frac{dX_r}{dt} &= f_{1x}(t)X_r + f_{1y}(t)Y_r + f_{1z}(t)Z_r + p_r(t), \\ \frac{dY_{r-1}}{dt} &= f_{2x}(t)X_r + f_{2y}(t)Y_r + f_{2z}(t)Z_r + q_r(t), \\ 0 &= f_{3x}(t)X_r + f_{3y}(t)Y_r + f_{3z}(t)Z_r + r_r(t), \\ X_r(0) &= \xi_r^*, \end{aligned} \tag{4.2$_r$}$$

where

$$\begin{aligned} f_{ix}(t) &= D_x f_i(X_0(t), Y_0(t), Z_0(t), 0), \\ f_{iy}(t) &= D_y f_i(X_0(t), Y_0(t), Z_0(t), 0), \qquad\qquad i = 1, 2, 3. \qquad (4.3) \\ f_{iz}(t) &= D_z f_i(X_0(t), Y_0(t), Z_0(t), 0), \end{aligned}$$

and the terms $p_r(t), q_r(t)$ and $r_r(t)$ are polynomials in $X_1, Y_1, Z_1, \ldots, X_{r-1}, Y_{r-1}, Z_{r-1}$ for which the coefficients are higher derivatives of the functions f_1, f_2, f_3 at the point $(X_0(t), Y_0(t), Z_0(t), 0)$. The right side ξ_r^* of the initial condition is the corresponding coefficient in the asymptotic expansion of $\xi^*(\epsilon)$ and p_r, q_r and r_r, $r = 0, 1, \ldots, N$, are obtained recursively. Therefore, $(p_r(t), q_r(t), r_r(t))$ is well defined on the interval $0 \le t \le T$ if the previous terms, $X_1, Y_1, Z_1, \ldots, X_{r-1}, Y_{r-1}, Z_{r-1}$, are well defined on $[0, T]$. A comparison of $(4.2)_r$ shows that, formally, all coefficient functions satisfy a linear system of the form

$$\begin{aligned} \frac{dx}{dt} &= a_{11}(t)x + a_{12}(t)y + a_{13}(t)z + b_1(t), \\ 0 &= a_{21}(t)x + a_{22}(t)y + a_{23}(t)z + b_2(t), \\ 0 &= a_{31}(t)x + a_{32}(t)y + a_{33}(t)z + b_3(t), \\ x(0) &= \xi, \end{aligned} \qquad (4.4)$$

for which

(a) a_{ij}, b_i are continuous on $[0, T]$;

(b) the matrix

$$A_1(t) = \begin{pmatrix} a_{22}(t) & a_{23}(t) \\ a_{32}(t) & a_{33}(t) \end{pmatrix}$$

is nonsingular for $t \in [0, T]$.

For such systems we obtain from Proposition 2.1 the existence result:

Proposition 4.1: *Under the assumptions (a) and (b) the system (4.4) has exactly one solution* $(x(t), y(t), z(t))$ *defined on the interval* $[0, T]$.

Proof: By assumption (b) we can solve the second and third equations in (4.4) for (y, z) in terms of x substitute into the first equation in (4.4). Thus, we obtain an initial value problem for a linear ODE. Then applying the basic existence theory for ODE's (see Theorem 5.2 in [CoLe]) we know that the system (4.4) possesses a unique solution. ∎

Since the informal expansion procedure provides that $p_r(t), q_r(t), r_r(t)$ are polynomials in $X_1, Y_1, Z_1, \ldots, X_{r-1}(t), Y_{r-1}(t), Z_{r-1}(t)$ with the coefficients depending on $(X_0(t), Y_0(t), Z_0(t))$, we can verify easily that the conditions (a) and (b) for the linear system (4.4) will hold for all systems $(4.2)_r$, $r = 1, \ldots, N$. Then by applying Proposition 2.1 to these systems $(4.2)_r$, we obtain the following existence theorem:

Proposition 4.2: *If the conditions in Proposition 3.1 hold then each system* $(4.2)_r$, $r = 1, \ldots, N$, *with any given value of* ξ_r^*, *has a unique solution* $(X_r(t), Y_r(t), Z_r(t))$ *defined on the domain* $[0, T]$ *of* $(X_0(t), Y_0(t), Z_0(t))$.

Proof: For any r, $1 \le r \le N$, the coefficient matrix in the system $(4.2)_r$,

$$\begin{pmatrix} f_{2y}(t) & f_{2z}(t) \\ f_{3y}(t) & f_{3z}(t) \end{pmatrix},$$

is nonsingular for all $t \in [0, T]$ due to the nonsingularity of $B(t)$. Hence by Proposition 4.1, we find that the system $(4.2)_r$ has a unique solution defined on the interval $[0, T]$. ∎

Note that, although the sequence of systems $(4.2)_r$ is derived formally under the hypothesis that their solutions $(X_r(t), Y_r(t), Z_r(t))$, $r = 0, 1, \ldots, N$, are the coefficients of the expansion series of an outer solution $(X(t,\epsilon), Y(t,\epsilon), Z(t,\epsilon))$ of (1.1a,b), these systems $(4.2)_r$ themselves are independent of the concept of an outer solution. So far, we only proved the existence and uniqueness of the systems $(4.2)_r$. Obviously, this does not mean the existence of an outer solution. But motivated by the procedure used in the derivation of the systems $(4.2)_r$, we may exploit the solutions $(X_r(t), Y_r(t), Z_r(t))$, $r = 0, 1, \ldots, N$, to construct an outer solution. This will be discussed in the next chapter.

5. Existence of outer solutions

This chapter concerns the existence of an outer solution of the outer problem (3.5a,b,c). We cite the following Lemma which plays an important role in the study of singularly perturbed ODE's.

Lemma 5.1: *Let* $A(t)$ *be an* n *by* n *continuous matrix for* $t_0 \le t \le t_1$ *and let the real parts of all its eigenvalues be less than* $-\mu$ *on* $t_0 \le t \le t_1$ *for some* $\mu > 0$. *Let* $\phi(t, s, \epsilon)$ *be the fundamental solution of*

$$\epsilon \frac{dX}{dt} = A(t)X$$
$$X(s) = I_{n \times n}$$

on $t_0 \le t \le t_1$ *for some* s *on* $t_0 \le s \le t_1$. *Then there exists a constant* K, *which is independent of* ϵ, *such that*

$$\|\phi(t, s, \epsilon)\| \le K e^{-\mu(t-s)/2\epsilon}$$

for $t_0 \le s \le t \le t_1$.

For the proof see,e.g. [Le].

For the theory, we require further assumptions about the solution $(X_0(t),\ Y_0(t),\ Z_0(t))$ of the reduced system (3.1a,b).

Assumption (III): *Proposition 3.1 holds and for any $t \in [0,T]$ all eigenvalues of the matrix $B(t)$, defined in (3.3b), remain strictly in the left half plane.*

In the previous chapter we formally derived the systems $(4.2)_r$ by assuming that an outer solution $(X(t,\epsilon), Y(t,\epsilon), Z(t,\epsilon))$ has an expansion of the form (4.1b), and we obtained existence and uniqueness results for solutions $(X_r(t), Y_r(t), Z_r(t))$ of $(2.3)_r$, $r = 1, \ldots, N$. The following theorem shows that these solutions $(X_r(t), Y_r(t), Z_r(t))$ can be used to construct an approximation of an outer solution $(X(t,\epsilon), Y(t,\epsilon), Z(t,\epsilon))$.

Theorem 5.1: *Under Assumptions (I) - (III) and for any given ξ_r^*, $r = 1, \ldots, N$, there is a constant $\epsilon_0 > 0$ ($\epsilon_0 \in \mathcal{J}_\epsilon$) such that the outer problem (3.5a,b,c) has for any ϵ, $0 < \epsilon \le \epsilon_0$ a solution $X = X(t,\epsilon)$, $Y = Y(t,\epsilon)$, $Z = Z(t,\epsilon) \in \mathcal{D}$, defined on the same interval $[0,T]$ as $(X_0(t), Y_0(t), Z_0(t))$, which satisfies*

$$
\begin{aligned}
X(t,\epsilon) - \sum_{i=0}^{N} X_i(t)\epsilon^i &= O(\epsilon^{N+1}), \\
Y(t,\epsilon) - \sum_{i=0}^{N} Y_i(t)\epsilon^i &= O(\epsilon^{N+1}), \qquad \text{as } \epsilon \to 0 \qquad (5.1)\\
Z(t,\epsilon) - \sum_{i=0}^{N} Z_i(t)\epsilon^i &= O(\epsilon^{N+1}),
\end{aligned}
$$

uniformly for $0 \le t \le T$.

[Proof] To begin the proof we simplify the outer problem (3.5) by introducing a change of variables defined by the affine mapping

$$\mathcal{T}_t : R^m \times R^n \times R^k \to R^m \times R^n \times R^k; \;\; \mathcal{T}_t(u,v,w) = (X,Y,Z), \qquad (5.2a)$$

$$
\begin{aligned}
X &= u + \sum_{r=0}^{N} X_r(t)\epsilon^r, \\
Y &= v + \sum_{r=0}^{N} Y_r(t)\epsilon^r + A_1(t)u, \qquad (5.2b)\\
Z &= w + \sum_{r=0}^{N} Z_r(t)\epsilon^r + A_2(t)u + B_1(t)v.
\end{aligned}
$$

Here A_1, A_2 and B_1 are chosen as

$$
\begin{aligned}
A_1(t) &= B(t)^{-1}(f_{2z}(t)(f_{3z}(t))^{-1} f_{3x}(t) - f_{2x}(t)), \\
A_2(t) &= -(f_{3z}(t))^{-1} f_{3y}(t) A_1(t) - (f_{3z}(t))^{-1} f_{3x}(t), \qquad (5.3)\\
B_1(t) &= -(f_{3z}(t))^{-1} f_{3y}(t),
\end{aligned}
$$

where $B(t)$ is defined in (3.3b). Since the domain of the functions $f_1(x,y,z,\epsilon)$, $f_2(x,y,z,\epsilon)$ and $f_3(x,y,z,\epsilon)$ is $\mathcal{D}\times\mathcal{J}_\epsilon$, we require that the new variables (u,v,w) remain in some suitable domain such that the range of the mapping $\mathcal{T}_t$ belongs to $\mathcal{D}$ for all $0\le t\le T$ and $0<\epsilon\le\epsilon_1$, where $\epsilon_1\in\mathcal{J}_\epsilon$ is sufficiently small. In order to find such a domain for $\mathcal{T}_t$ it is important to note that $(X_0(t),Y_0(t),Z_0(t))$ is an interior point in $\mathcal{D}$ for any $t\in[0,T]$. In fact, since (5.2b) can be written as

$$
\begin{aligned}
X &= X_0(t)+u+\left(\sum_{r=1}^{N}X_r(t)\epsilon^{r-1}\right)\epsilon,\\
Y &= Y_0(t)+v+A_1(t)u+\left(\sum_{r=1}^{N}Y_r(t)\epsilon^{r-1}\right)\epsilon,\\
Z &= Z_0(t)+A_2(t)u+B_1(t)v+w+\left(\sum_{r=1}^{N}Z_r(t)\epsilon^{r-1}\right)\epsilon.
\end{aligned}\tag{5.4}
$$

and $(X_r(t),Y_r(t),Z_r(t))$, $r=1,\ldots,N$, $A_1(t)$, $A_2(t)$ and $B_1(t)$ are uniformly bounded in t, $0\le t\le T$, it follows from $(X_0(t),Y_0(t),\ Z_0(t))\in int\ (\mathcal{D})\ \forall t\in[0,T]$ that there exist positive numbers ϵ_1 $(\le\epsilon_0\in\mathcal{J}_\epsilon)$ and δ such that $\mathcal{T}_t(u,v,w)\in\mathcal{D}$ for any given t, $t\in[0,T]$ and for all $(u,v,w)\in B(0,\delta)\times B(0,\delta)\times B(0,\delta)$. Accordingly, for

$$
\begin{gathered}
\hat{\mathcal{D}}=\mathcal{D}_u\times\mathcal{D}_v\times\mathcal{D}_w,\\
\mathcal{D}_u=B(0,\delta)\subset R^m,\quad \mathcal{D}_v=B(0,\delta)\subset R^n,\quad \mathcal{D}_w=B(0,\delta)\subset R^k,
\end{gathered}
$$

we have $\mathcal{T}_t(\hat{\mathcal{D}})\subset\mathcal{D}$ for all $0\le t\le T$.

Substituting (5.2b) into the first equation of (3.5a) we obtain that

$$
\begin{aligned}
&\frac{du}{dt}+\sum_{i=0}^{N}\frac{dX_i(t)}{dt}\epsilon^i\\
&=f_1(X_0(t)+u+\sum_{i=1}^{N},Y_0(t)+v+A_1(t)u+\sum_{i=1}^{N}Y_i(t)\epsilon^i,\\
&\quad Z_0(t)+w+A_2(t)u+B_1(t)v+\sum_{i=1}^{N}Z_i(t)\epsilon^i,\epsilon).
\end{aligned}\tag{5.5}
$$

We introduce the Taylor expansion of f_1 at $(X_0(t),\ Y_0(t),\ Z_0(t),0)$. For this the abbreviations

will be used

$$\mathcal{A}(t) = \begin{pmatrix} I & 0 & 0 & 0 \\ A_1(t) & I & 0 & 0 \\ A_2(t) & B_1(t) & I & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\Omega_0(t) = \begin{pmatrix} X_0(t) \\ Y_0(t) \\ Z_0(t) \\ 0 \end{pmatrix}, \quad \mathcal{U} = \begin{pmatrix} u \\ v \\ w \\ 0 \end{pmatrix}, \quad \Omega_i(t) = \begin{pmatrix} X_i(t) \\ Y_i(t) \\ Z_i(t) \\ \delta_{i1} \end{pmatrix}, \quad i = 1, \ldots, N, \tag{5.6}$$

$$\Omega = \begin{pmatrix} X \\ Y \\ Z \\ \epsilon \end{pmatrix} = \Omega_0(t) + \mathcal{A}(t)\mathcal{U} + \sum_{i=1}^{N} \Omega_i(t)\epsilon^i,$$

where δ_{kl} is the Kronecker delta. Then the expansion of $f_1(\Omega)$ at $\Omega_0(t)$ has the form

$$\begin{aligned} &f_1(\Omega) \\ &= f_1(\Omega_0(t)) + \sum_{r=1}^{N} \frac{1}{r!} D^{(r)} f_1(\Omega_0(t)) (\sum_{i=1}^{N} \Omega_i(t)\epsilon^i)^r + Df_1(\Omega_0(t))\mathcal{A}(t)\mathcal{U} + \\ &\sum_{r=2}^{N} \frac{1}{r!} D^{(r)} f_1(\Omega_0(t)) [(\mathcal{A}(t)\mathcal{U} + \sum_{i=1}^{N} \Omega_i(t)\epsilon^i)^r - (\sum_{i=1}^{N} \Omega_i(t)\epsilon^i)^r] + G(t, u, v, w, \epsilon) \end{aligned} \tag{5.7}$$

where $G(t, u, v, w, \epsilon)$ is the remainder term. Obviously, $G(t, u, v, w, \epsilon)$ satisfies the following **Condition (N)**

Definition 5.1: *A C^1-function $F(t, u, v, w, \epsilon) : R \times R^m \times R^n \times R^k \times R \to R^s$ is said to satisfy the* **Condition (N)** *if it satisfies the following asymptotic relations:*

(i)

$$F(t, 0, 0, 0, \epsilon) = O(\epsilon^N), \qquad \text{as } \epsilon \to 0;$$

uniformly for $0 \le t \le T$, where N is a positive integer;

(ii)

$$\left.\begin{aligned} D_u F(t, u, v, w, \epsilon) &= O(\epsilon + |u| + |v| + |w|) \\ D_v F(t, u, v, w, \epsilon) &= O(\epsilon + |u| + |v| + |w|) \\ D_w F(t, u, v, w, \epsilon) &= O(\epsilon + |u| + |v| + |w|) \end{aligned}\right\} \quad \text{as } \epsilon, |u|, |v|, |w| \to 0$$

uniformly for $0 \le t \le T$.

It is noticed that the Condition (N) is the same conditions as (1.2a) and (1.2b) in Hypothesis (H) in [Ya2]. Observe that

$$\begin{aligned} &Df_1(\Omega_0(t))\mathcal{U} \\ &= D_x f_1(\Omega_0(t))u + D_y f_1(\Omega_0(t))(v + A_1(t)u) + D_z f_1(\Omega_0(t))(w + A_2(t)u + B_1(t)v) \\ &= (D_x f_1(\Omega_0(t)) + D_y f_1(\Omega_0(t))A_1(t) + D_z f_1(\Omega_0(t))A_2(t))u + \\ &\quad (D_y f_1(\Omega_0(t)) + D_z f_1(\Omega_0(t)B_1(t))v + D_z f_1(\Omega_0(t))w \end{aligned} \tag{5.8}$$

and

$$\sum_{r=1}^{N} \frac{1}{r!} D^{(r)} f_1(\Omega_0(t)) (\sum_{i=1}^{N} \Omega_i(t)\epsilon^i)^r$$
$$= \sum_{r=1}^{N} \frac{1}{r!} D^{(r)} f_1(\Omega_0(t)) \sum_{k_1,\ldots,k_r=1}^{N} \left(\prod_{j=1}^{r} \Omega_{k_j}(t) \right) \epsilon^{k_1+\cdots+k_r}$$
$$= \sum_{s=1}^{N} b_s(t)\epsilon^s + R_N(t,\epsilon)$$

where

$$b_s(t) = \sum_{r=1}^{N} \frac{1}{r!} D^{(r)} f_1(\Omega_0(t)) \sum_{\substack{k_1+\cdots+k_r=s \\ 1\le k_i \le s}} \left(\prod_{j=1}^{r} \Omega_{k_j}(t) \right)$$

and $R_N(t,\epsilon) = O(\epsilon^{N+1})$ is independent of (u,v,w). Since $(X_s(t), Y_s(t), Z_s(t))$, $s = 0,1,\ldots,N$ satisfy

$$\frac{dX_s}{dt} = b_s(t)$$

where $b_0(t) = f_1(\Omega_0(t))$, it follows from (5.5,6,7) that

$$\begin{aligned}
\frac{du}{dt} &= (D_x f_1(\Omega_0(t)) + D_y f_1(\Omega_0(t))A_1(t) + D_z f_1(\Omega_0(t))A_2(t))u + \\
&\quad (D_y f_1(\Omega_0(t)) + D_z f_1(\Omega_0(t)B_1(t))v + D_z f_1(\Omega_0(t))w + R_N(t,\epsilon) \\
&\quad + \sum_{r=2}^{N} \frac{1}{r!} D^{(r)} f_1(\Omega_0(t))[(\mathcal{A}(t)\mathcal{U} + \sum_{i=1}^{N} \Omega_i(t)\epsilon^i)^r - (\sum_{i=1}^{N} \Omega_i(t)\epsilon^i)^r] \\
&\quad + G(t,u,v,w,\epsilon) \\
&= C_1(t)u + L_1(t)v + E_1(t)w + \hat{F}_1(t,u,v,w,\epsilon)
\end{aligned} \tag{5.9}$$

where

$$\begin{aligned}
C_1(t) &= D_x f_1(\Omega_0(t)) + D_y f_1(\Omega_0(t))A_1(t) + D_z f_1(\Omega_0(t))A_2(t), \\
L_1(t) &= D_y f_1(\Omega_0(t)) + D_z f_1(\Omega_0(t)B_1(t), \\
E_1(t) &= D_z f_1(\Omega_0(t)), \\
\hat{F}_1(t,u,v,w,\epsilon) &= \sum_{r=2}^{N} \frac{1}{r!} D^{(r)} f_1(\Omega_0(t))[(\mathcal{A}(t)\mathcal{U} + \sum_{i=1}^{N} \Omega_i(t)\epsilon^i)^r - (\sum_{i=1}^{N} \Omega_i(t)\epsilon^i)^r] \\
&\quad + R_N(t,\epsilon) + G(t,u,v,w,\epsilon).
\end{aligned} \tag{5.10}$$

Obviously

$$R_N(t,\epsilon)$$

and

$$\sum_{r=2}^{N} \frac{1}{r!} D^{(r)} f_1(\Omega_0(t))[(\mathcal{A}(t)\mathcal{U} + \sum_{i=1}^{N} \Omega_i(t)\epsilon^i)^r - (\sum_{i=1}^{N} \Omega_i(t)\epsilon^i)^r]$$

satisfy **Condition (N)**. Since also G satisfies **Condition (N)**, this implies that $\hat{F}_1$ satisfies **Condition (N)** as well.

Now substitute (5.2b) into the second and third equations of (3.5a) and expand f_2, f_3 at $(X_0(t), Y_0(t), Z_0(t), 0)$. Then, because $(X_i(t), Y_i(t), Z_i(t))$, $i = 0, 1, \dots, N$, are solutions of $(4.2)_i$, we obtain two further equations which, with (5.9) form the the following system equivalent to (3.5) under the mapping $\mathcal{T}_t$:

$$\begin{aligned}
\frac{du}{dt} &= C_1(t)u + L_1(t)v + E_1(t)w + \hat{F}_1(t, u, v, w, \epsilon) \\
\epsilon \frac{dv}{dt} &= \qquad\qquad\quad B(t)v + E_2(t)w + \hat{F}_2(t, u, v, w, \epsilon) \\
0 &= \qquad\qquad\qquad\qquad\quad E_3(t)w + \hat{F}_3(t, u, v, w, \epsilon) \\
u(0, \epsilon) &= \xi^*(\epsilon) - \sum_{r=0}^{N} \xi_r^* \epsilon^r = \theta_N(\epsilon) = O(\epsilon^{N+1})
\end{aligned} \tag{5.11}$$

Here

$$E_2(t) = D_z f_2(\Omega_0(t)), \qquad E_3(t) = D_z f_3(\Omega_0(t),$$

and $\hat{F}_2$ and $\hat{F}_3$ satisfy **Condition (N)**. Note that in the derivation of the last two equations of (5.11), we used the fact that

$$\begin{aligned}
&D_x f_2(\Omega_0(t)) + D_y f_2(\Omega_0(t))A_1(t) + D_z f_2(\Omega_0(t))A_2(t) = 0, \\
&D_x f_3(\Omega_0(t)) + D_y f_3(\Omega_0(t))A_1(t) + D_z f_3(\Omega_0(t))A_2(t) = 0, \\
&D_y f_3(\Omega_0(t)) + D_z f_3(\Omega_0(t))B_1(t) = 0,
\end{aligned} \tag{5.12}$$

and

$$D_y f_2(\Omega_0(t)) + D_z f_2(\Omega_0(t))B_1(t) = B(t)$$

where $B(t)$ is defined in (3.3b).With the notations (5.6) this is equivalent with

$$\begin{aligned}
Df_2(\Omega_0(t))\mathcal{A}(t) &= (0, B(t), D_z f_2(\Omega_0(t)), D_\epsilon f_2(\Omega_0(t))), \\
Df_3(\Omega_0(t))\mathcal{A}(t) &= (0, 0, D_z f_3(\Omega_0(t)), D_\epsilon f_3(\Omega_0(t))).
\end{aligned}$$

In other words, from the defining relation (5.3) we find that $A_1(t), A_2(t)$ and $B_1(t)$ solve (5.12).

If we can show that there exists a solution $(u(t, \epsilon), v(t, \epsilon), w(t, \epsilon)) \in \hat{\mathcal{D}}$ of the system (5.11) which satisfies

$$u(t, \epsilon) = O(\epsilon^{N+1}), \;\; v(t, \epsilon) = O(\epsilon^{N+1}), \;\; w(t, \epsilon) = O(\epsilon^{N+1}), \quad \text{as } \epsilon \to 0 \tag{5.13}$$

uniformly for $0 \le t \le T$, then Theorem 5.1 is proved. But note that we need only the existence of a solution of (3.5) without requiring its uniqueness. Hence instead of (5.11) we consider the constrained system of integral equations

$$
\begin{aligned}
u(t,\epsilon) &= \Phi(t)(\theta_N(\epsilon) + \int_0^t \Phi^{-1}(s)(L_1(s)v(s,\epsilon) + E_1(s)w(s,\epsilon) \\
&\qquad + \hat{F}_1(s,u(s,\epsilon),v(s,\epsilon),w(s,\epsilon),\epsilon))ds) \\
v(t,\epsilon) &= \int_0^t \frac{\Psi(t,s,\epsilon)}{\epsilon}(E_2(s)w(s,\epsilon) + \hat{F}_2(s,u(s,\epsilon),v(s,\epsilon),w(s,\epsilon),\epsilon))ds \\
w(t,\epsilon) &= -(E_3(t))^{-1}\hat{F}_3(t,u(t,\epsilon),v(t,\epsilon),w(t,\epsilon),\epsilon)
\end{aligned} \tag{5.14}
$$

where $\Phi(t)$ satisfies

$$
\begin{aligned}
\frac{d\Phi(t)}{dt} &= C_1(t)\Phi(t), \qquad 0 \le t \le T \\
\Phi(0) &= I
\end{aligned}
$$

while $\Psi(t,s,\epsilon)$ is the solution of the following system:

$$
\begin{aligned}
\frac{d\Psi}{dt} &= \frac{1}{\epsilon}B(t)\Psi, \qquad 0 \le s \le t \le T \\
\Psi(t,s,\epsilon)|_{t=s} &= I.
\end{aligned}
$$

Note that the system (5.14) is not equivalent with the system (5.11). But, obviously, if $(u(t,\epsilon)$, $v(t,\epsilon)$, $w(t,\epsilon)) \in \hat{\mathcal{D}}$ solves (5.14), then it solves (5.11) as well. Thus if we can prove that (5.14) admits a solution which satisfies the asymptotic relation (5.13), then we are done. We will use the theorems in [Ya2] to prove this existence.

By Assumption (III) there exists a positive number μ such that

$$\Re(\lambda(B(t))) \le -\mu$$

for all eigenvalues of $B(t)$, $0 \le t \le T$. Then Lemma 5.1 ensures the existence of a constant K, which is independent of ϵ, and such that

$$|\Psi(t,s,\epsilon)| \le Ke^{-\frac{\mu(t-s)}{2\epsilon}}, \quad 0 \le s \le t \le T.$$

With the notations

$$
\begin{aligned}
&\theta_1(t,\epsilon) = \Phi(t)\theta_N(\epsilon), \;\; \theta_2(t,\epsilon) = 0, \;\; G_1(t,s,\epsilon) = \Phi(t)\Phi^{-1}(s)L_1(s), \\
&H_1(t,s,\epsilon) = \Phi(t)\Phi^{-1}(s)E_1(s), \;\; K(t,s,\epsilon) = \Psi(t,s,\epsilon)E_2(s), \\
&F_1 = \Phi(t)\Phi^{-1}(s)\hat{F}_1, \;\; F_2 = \hat{F}_2, \;\; F_3 = -E_3^{-1}(t)\hat{F}_3,
\end{aligned} \tag{5.15}
$$

(5.14) can be written in the form of constrained systems of integral equations as the one in [Ya2], and evidently the Hypothesis (H) in [Ya2] are satisfied. Therefore by the results of

Theorem 1, 2 in [Ya2], (4.13) has a unique solution $(u(t,\epsilon), v(t,\epsilon), w(t,\epsilon)) \in \hat{\mathcal{D}}$, for all $t \in [0,T]$ and $0 < \epsilon \leq \epsilon_2$ ($\leq \epsilon_1$) that satisfies (4.12). Therefore, under the transformation (4.1), $(X(t,\epsilon)$, $Y(t,\epsilon)$, $Z(t,\epsilon))$ remains in $\mathcal{D}$ for all $t \in [0,T]$ and $0 < \epsilon \leq \epsilon_2$ and is an outer solution which satisfies the asymptotic relations (4.1b). This completes the proof of Theorem 5.1. ∎

6. Asymptotic expansion of inner solutions

6.1. Formal derivation of asymptotic expansion of inner solutions. Now to proceed to the inner system, by using **Assumption II**, for the initial conditions $\alpha(0,\epsilon)$, $\beta(0,\epsilon)$, $\gamma(0,\epsilon)$ in (3.7a) we must have

$$\begin{aligned}
\alpha(0,\epsilon) &= \xi(\epsilon) - \xi^*(\epsilon) = \sum_{i=1}^{N}(\xi_i - \xi_i^*)\epsilon^i + O(\epsilon^{N+1}),\\
\beta(0,\epsilon) &= \eta(\epsilon) - Y(0,\epsilon) = \sum_{i=0}^{N}(\eta_i - Y_i(0))\epsilon^i + O(\epsilon^{N+1}),\\
\gamma(0,\epsilon) &= \zeta(\epsilon) - Z(0,\epsilon) = \sum_{i=0}^{N}(\zeta_i - Z_i(0))\epsilon^i + O(\epsilon^{N+1}).
\end{aligned} \tag{6.1}$$

Then, formally, we expect the solutions of (3.7a) to satisfy asymptotic relations of the form

$$\begin{aligned}
\alpha(\tau,\epsilon) &= \sum_{i=0}^{N}\alpha_i(\tau)\epsilon^i + O(\epsilon^{N+1})\\
\beta(\tau,\epsilon) &= \sum_{i=0}^{N}\beta_i(\tau)\epsilon^i + O(\epsilon^{N+1})\\
\gamma(\tau,\epsilon) &= \sum_{i=0}^{N}\gamma_i(\tau)\epsilon^i + O(\epsilon^{N+1})
\end{aligned} \tag{6.2}$$

which should hold uniformly for $0 \leq \tau \leq T/\epsilon$.

We will substitute (6.2) into (3.7a), expand right sides at $(X_0(0) + \alpha_0(\tau)$, $Y_0(0) + \beta_0(\tau)$, $Z_0(0) + \gamma_0(\tau))$ and collect terms with equal powers of ϵ to obtain the equations which $(\alpha_i(\tau)$, $\beta_i(\tau)$, $\gamma_i(\tau))$ must satisfy. Since the expansions of the right sides in (3.7a) involve very tedious derivations, we set for abbreviation

$$\Omega(\tau,\epsilon) = \begin{pmatrix} \alpha(\tau,\epsilon) + X(\epsilon\tau,\epsilon)\\ \beta(\tau,\epsilon) + Y(\epsilon\tau,\epsilon)\\ \gamma(\tau,\epsilon) + Z(\epsilon\tau,\epsilon)\\ \epsilon \end{pmatrix}, \quad \Gamma(\tau,\epsilon) = \begin{pmatrix} X(\epsilon\tau,\epsilon)\\ Y(\epsilon\tau,\epsilon)\\ Z(\epsilon\tau,\epsilon)\\ \epsilon \end{pmatrix}.$$

Then

$$\hat{f}_i(\epsilon\tau, \alpha(\tau,\epsilon), \beta(\tau,\epsilon), \gamma(\tau,\epsilon), \epsilon) = f_i(\Omega(\tau,\epsilon)) - f_i(\Gamma(\tau,\epsilon)), \quad i = 1,2,3.$$

For simplicity, in the following, the subscript i will be omitted.

For later use we state the following two identities:

$$\sum_{\substack{i=1\\j=0}}^{N} a_{ij}\epsilon^{i+j} = \sum_{k=1}^{N}\left(\sum_{i=1}^{k} a_{ik-i}\right)\epsilon^k + O(\epsilon^{N+1}) \tag{6.3}$$

and

$$h_n(\epsilon) \equiv \left(\sum_{k=1}^{N} a_k\epsilon^k\right)^n = \sum_{k_1,\ldots,k_n=1}^{N} \prod_{s=1}^{n} a_{k_s}\epsilon^{k_1+\cdots+k_n}. \tag{6.4}$$

From (6.4) it follows that in the polynomial $\sum_{n=1}^{N} d_n h_n(\epsilon)$ in ϵ the coefficient of ϵ^r is

$$\sum_{n=1}^{r} d_n \sum_{\substack{k_1+\cdots+k_n=r\\1\le k_i\le r}} \prod_{s=1}^{n} a_{k_s}, \quad \text{if } 0 \le r \le N \tag{6.5}$$

while for r exceeding N it is

$$\sum_{n=1}^{N} d_n \sum_{\substack{k_1+\cdots+k_n=r\\1\le k_i\le N}} \prod_{s=1}^{n} a_{k_s}. \tag{6.6}$$

Since $(X(t,\epsilon),\ Y(t,\epsilon),\ Z(t,\epsilon))$ and $(\alpha(\tau,\epsilon),\ \beta(\tau,\epsilon),\ \gamma(\tau,\epsilon))$ satisfy the asymptotic relations (4.1a,b) and (6.2), respectively, we obtain, with (6.3), that

$$\begin{aligned}
&X(\epsilon\tau,\epsilon) + \alpha(\tau,\epsilon)\\
&= X_0(\epsilon\tau) + \alpha_0(\tau) + \sum_{i=1}^{N}(X_i(\epsilon\tau) + \alpha_i(\tau))\epsilon^i + O(\epsilon^{N+1})\\
&= X_0(0) + \alpha_0(\tau) + \sum_{k=1}^{N}\left(\alpha_k(\tau) + \sum_{i=0}^{k}\frac{X_i^{(k-i)}(0)\tau^{k-i}}{(k-i)!}\right)\epsilon^k + O(\epsilon^{N+1}),
\end{aligned} \tag{6.7}$$

and similarly that

$$X(\epsilon\tau,\epsilon) = X_0(0) + \sum_{k=1}^{N}\left(\sum_{i=0}^{k}\frac{X_i^{(k-i)}(0)\tau^{k-i}}{(k-i)!}\right)\epsilon^k + O(\epsilon^{N+1}). \tag{6.8}$$

With (6.7), (6.8) and the analogous expansions for $Y(\epsilon\tau,\epsilon)+\alpha(\tau,\epsilon)$, $Y(\epsilon\tau,\epsilon)$, $Z(\epsilon\tau,\epsilon)+\alpha(\tau,\epsilon)$ and $Z(\epsilon\tau,\epsilon)$, we find that

$$\begin{aligned}
\Omega(\tau,\epsilon) &= \Omega_0(\tau) + \sum_{k=1}^{N}\Omega_k(\tau)\epsilon^k + (\epsilon^{N+1}),\\
\Gamma(\tau,\epsilon) &= \Gamma_0 + \sum_{k=1}^{N}\Gamma_k(\tau)\epsilon^k + (\epsilon^{N+1})
\end{aligned} \tag{6.9}$$

where

$$\Omega_0(\tau) = \begin{pmatrix} X_0(0) + \alpha_0(\tau) \\ Y_0(0) + \beta_0(\tau) \\ Z_0(0) + \gamma_0(\tau) \\ 0 \end{pmatrix}, \ \Omega_k(\tau) = \begin{pmatrix} \alpha_k(\tau) + \sum_{i=0}^{k} \frac{X_i^{(k-i)}(0)\tau^{k-i}}{(k-i)!} \\ \beta_k(\tau) + \sum_{i=0}^{k} \frac{Y_i^{(k-i)}(0)\tau^{k-i}}{(k-i)!} \\ \gamma_k(\tau) + \sum_{i=0}^{k} \frac{Z_i^{(k-i)}(0)\tau^{k-i}}{(k-i)!} \\ \delta_{k1} \end{pmatrix}, \tag{6.10}$$

$$\Gamma_0 = \begin{pmatrix} X_0(0) \\ Y_0(0) \\ Z_0(0) \\ 0 \end{pmatrix}, \ \Gamma_k(\tau) = \begin{pmatrix} \sum_{i=0}^{k} \frac{X_i^{(k-i)}(0)\tau^{k-i}}{(k-i)!} \\ \sum_{i=0}^{k} \frac{Y_i^{(k-i)}(0)\tau^{k-i}}{(k-i)!} \\ \sum_{i=0}^{k} \frac{Z_i^{(k-i)}(0)\tau^{k-i}}{(k-i)!} \\ \delta_{k1} \end{pmatrix}. \tag{6.11}$$

and δ_{kl} is the Kronecker delta. It follows from (6.10) and (6.11) that

$$\begin{aligned} &\hat{f}(\epsilon\tau, \alpha(\tau,\epsilon), \beta(\tau,\epsilon), \gamma(\tau,\epsilon), \epsilon) = f(\Omega(\tau,\epsilon)) - f(\Gamma(\tau,\epsilon)) \\ &= f(\Omega_0(\tau) + \sum_{k=1}^{N} \Omega_k(\tau)\epsilon^k + O(\epsilon^{N+1})) - f(\Gamma_0 + \sum_{k=1}^{N} \Gamma_k(\tau)\epsilon^k + O(\epsilon^{N+1})) \\ &= f(\Omega_0(\tau)) + \sum_{j=1}^{N} \frac{D^j f(\Omega_0(\tau))}{j!} \left(\sum_{k=1}^{N} \Omega_k(\tau)\epsilon^k \right)^j \\ &\quad - f(\Gamma_0) - \sum_{j=1}^{N} \frac{D^j f(\Gamma_0)}{j!} \left(\sum_{k=1}^{N} \Gamma_k(\tau)\epsilon^k \right)^j + O(\epsilon^{N+1}). \end{aligned} \tag{6.12}$$

By collecting equal powers of ϵ in (6.12), with (6.6) we find that the coefficient $c_r(\tau)$ of ϵ^r, $1 \le r \le N$, is:

$$\begin{aligned} c_r(\tau) &= \sum_{j=1}^{r} \frac{D^j f(\Omega_0(\tau))}{j!} \sum_{\substack{k_1+\cdots+k_j=r \\ 1\le k_j \le r}} \prod_{s=1}^{j} \Omega_{k_s}(\tau) - \sum_{j=1}^{r} \frac{D^j f(\Gamma_0)}{j!} \sum_{\substack{k_1+\cdots+k_j=r \\ 1\le k_j \le r}} \prod_{s=1}^{j} \Gamma_{k_s}(\tau) \\ &= Df(\Omega_0(\tau))\Omega_r(\tau) + \sum_{j=2}^{r} \frac{D^j f(\Omega_0(\tau))}{j!} \sum_{\substack{k_1+\cdots+k_j=r \\ 1\le k_j < r}} \prod_{s=1}^{j} \Omega_{k_s}(\tau) \\ &\quad - \sum_{j=1}^{r} \frac{D^j f(\Gamma_0)}{j!} \sum_{\substack{k_1+\cdots+k_j=r \\ 1\le k_j \le r}} \prod_{s=1}^{j} \Gamma_{k_s}(\tau). \end{aligned} \tag{6.13}$$

Since, by (6.10),

$$
\begin{aligned}
&Df(\Omega_0(\tau))\Omega_r(\tau) = D_x f(\Omega_0(\tau))\alpha_r(\tau) + D_y f(\Omega_0(\tau))\beta_r(\tau) + D_z f(\Omega_0(\tau))\gamma_r(\tau) \\
&\quad + D_x f(\Omega_0(\tau)) \sum_{i=0}^{r} \frac{X_i^{(r-i)}(0)\tau^{k-i}}{(k-i)!} + D_y f(\Omega_0(\tau)) \sum_{i=0}^{r} \frac{Y_i^{(r-i)}(0)\tau^{k-i}}{(k-i)!} \\
&\quad + D_z f(\Omega_0(\tau)) \sum_{i=0}^{r} \frac{Z_i^{(r-i)}(0)\tau^{k-i}}{(k-i)!} + D_\epsilon f(\Omega_0(\tau))\delta_{r1},
\end{aligned} \tag{6.14}
$$

it follows from (6.13) that

$$
c_r(\tau) = D_x f(\Omega_0(\tau))\alpha_r(\tau) + D_y f(\Omega_0(\tau))\beta_r(\tau) + D_z f(\Omega_0(\tau))\gamma_r(\tau) + P_r(\tau) \tag{6.15}
$$

where

$$
\begin{aligned}
&P_r(\tau) \\
&\quad = D_x f(\Omega_0(\tau)) \sum_{i=0}^{r} \frac{X_i^{(r-i)}(0)\tau^{k-i}}{(k-i)!} + D_y f(\Omega_0(\tau)) \sum_{i=0}^{r} \frac{Y_i^{(r-i)}(0)\tau^{k-i}}{(k-i)!} \\
&\quad + D_z f(\Omega_0(\tau)) \sum_{i=0}^{r} \frac{Z_i^{(r-i)}(0)\tau^{k-i}}{(k-i)!} + D_\epsilon f(\Omega_0(\tau))\delta_{r1} \\
&\quad + \sum_{j=2}^{r} \frac{D^j f(\Omega_0(\tau))}{j!} \sum_{\substack{k_1+\cdots+k_j=r \\ 1\le k_j<r}} \prod_{s=1}^{j} \Omega_{k_s}(\tau) - \sum_{j=1}^{r} \frac{D^j f(\Gamma_0)}{j!} \sum_{\substack{k_1+\cdots+k_j=r \\ 1\le k_j\le r}} \prod_{s=1}^{j} \Gamma_{k_s}(\tau) \\
&= (D_\epsilon f(\Omega_0(\tau)) - D_\epsilon f(\Gamma_0))\delta_{r1} + (D_x f(\Omega_0(\tau)) - D_x f(\Gamma_0)) \sum_{i=0}^{r} \frac{X_i^{(r-i)}(0)\tau^{k-i}}{(k-i)!} \\
&\quad + (D_y f(\Omega_0(\tau)) - D_y f(\Gamma_0)) \sum_{i=0}^{r} \frac{Y_i^{(r-i)}(0)\tau^{k-i}}{(k-i)!} \\
&\quad + (D_z f(\Omega_0(\tau)) - D_z f(\Gamma_0)) \sum_{i=0}^{r} \frac{Z_i^{(r-i)}(0)\tau^{k-i}}{(k-i)!} \\
&\quad + \sum_{j=2}^{r} \frac{D^j f(\Omega_0(\tau))}{j!} \sum_{\substack{k_1+\cdots+k_j=r \\ 1\le k_j<r}} \prod_{s=1}^{j} \Omega_{k_s}(\tau) - \sum_{j=2}^{r} \frac{D^j f(\Gamma_0)}{j!} \sum_{\substack{k_1+\cdots+k_j=r \\ 1\le k_j<r}} \prod_{s=1}^{j} \Gamma_{k_s}(\tau)
\end{aligned} \tag{6.16a}
$$

which is a polynomial in $\alpha_1, \beta_1, \gamma_1, \ldots, \alpha_{r-1}, \beta_{r-1}, \gamma_{r-1}$ with the coefficients depending on $D^j f(\Omega_0(\tau))$, $j = 1, \ldots, r$ and $D^j f(\Gamma_0)$, $(X_j^{(i)}(0),\ Y_j^{(i)}(0),\ Z_j^{(i)}(0))$, $i, j = 0, 1, \ldots, r$. The constant term in the polynomial $P_r(\tau)$ is

$$
\begin{aligned}
&\sum_{j=1}^{r} \left(D^j f(\Omega_0(\tau)) - D^j f(\Gamma_0)\right)/j! \sum_{\substack{k_1+\cdots+k_j=r \\ 1\le k_j\le r}} \prod_{s=1}^{j} \Gamma_{k_s}(\tau) \\
&= \sum_{j=1}^{r} \left(D^j f(\Omega_0(\tau)) - D^j f(\Gamma_0)\right) p_{rj}(\tau)
\end{aligned} \tag{6.16b}
$$

where

$$p_{rj}(\tau) = \frac{1}{j!} \sum_{\substack{k_1+\cdots+k_j=r \\ 1\le k_j \le r}} \prod_{s=1}^{j} \Gamma_{k_s}(\tau) \tag{6.16c}$$

Obviously, by (6.11), the degree of the polynomial $p_{rj}(\tau)$ in τ is $\le r$. In other words, the constant term in the polynomial P_r is the sum of certain terms, each of which is a product of the factor of the difference between the derivatives of f at $\Omega_0(\tau)$ and Γ_0 and the polynomial $p_{rj}(\tau)$ in τ with degree $\le r$.

Now substituting (6.2) into (3.7a), with (6.15) we obtain that

$$\begin{aligned}
&\sum_{r=0}^{N} \frac{d\alpha_r(\tau)}{d\tau}\epsilon^r + O(\epsilon^{N+1}) \\
&= \epsilon(f_1(\Omega(\tau,\epsilon)) - f_1(\Gamma(\tau,\epsilon))) \\
&= \epsilon\{f_1(\Omega_0(\tau)) - f_1(\Gamma_0(\tau)) + \sum_{r=1}^{N} c_r^{(1)}(\tau)\epsilon^r\} + O(\epsilon^{N+1}),
\end{aligned}$$

$$\begin{aligned}
&\sum_{r=0}^{N} \frac{d\beta_r(\tau)}{d\tau}\epsilon^r + O(\epsilon^{N+1}) \\
&= f_2(\Omega_0(\tau)) - f_2(\Gamma_0) + \sum_{r=1}^{N} c_r^{(2)}(\tau)\epsilon^r + O(\epsilon^{N+1}),
\end{aligned}$$

$$0 = f_3(\Omega_0(\tau)) - f_3(\Gamma_0) + \sum_{r=1}^{N} c_r^{(3)}(\tau)\epsilon^r + O(\epsilon^{N+1}),$$

where $c_r^{(i)}(\tau)$ is defined in (6.15) with f replaced by f_i, $i = 1, 2, 3$. Since

$$\begin{aligned}
f_2(\Gamma_0) &= f_2(X_0(0), Y_0(0), Z_0(0), 0) = 0, \\
f_3(\Gamma_0) &= f_3(X_0(0), Y_0(0), Z_0(0), 0) = 0,
\end{aligned}$$

we find, by collecting terms with equal powers of ϵ, that $\alpha_0(\tau) \equiv 0$, because $\frac{d\alpha_0}{d\tau} = 0$, $\alpha_0(0) = 0$, and (since $X_0(0) = \xi_0$,)

$$\begin{aligned}
\frac{d\beta_0(\tau)}{d\tau} &= f_2(\xi_0, Y_0(0) + \beta_0, Z_0(0) + \gamma_0, 0), \\
0 &= f_3(\xi_0, Y_0(0) + \beta_0, Z_0(0) + \gamma_0, 0), \\
\beta_0(0) &= \eta_0 - Y_0(0), \gamma_0(0) = \zeta_0 - Z_0(0),
\end{aligned} \tag{6.17$_0$}$$

and, generally,

$$\begin{aligned}
\frac{d\alpha_r(\tau)}{d\tau} &= P_{\alpha,r}(\tau), \\
\frac{d\beta_r(\tau)}{d\tau} &= D_x f_2(\Omega_0(\tau))\alpha_r + D_y f_2(\Omega_0(\tau))\beta_r + D_z f_2(\Omega_0((\tau))\gamma_r + P_{\beta,r}(\tau), \\
0 &= D_x f_3(\Omega_0(\tau))\alpha_r + D_y f_3(\Omega_0(\tau))\beta_r + D_z f_3(\Omega_0(\tau))\gamma_r + P_{\gamma,r}(\tau), \\
\alpha_r(0) &= \xi_r - \xi_r^*, \beta_r(0) = \eta_r - Y_r(0), \gamma_r(0) = \zeta_r - Z_r(0)
\end{aligned} \tag{6.17$_r$}$$

for $r = 1, \ldots, N$, where $P_{\alpha,r}(\tau) = c_{r-1}^{(1)}(\tau)$, and $P_{\beta,r}(\tau)$, $P_{\gamma,r}(\tau)$ are defined by (6.16a,b,c) with f replaced by f_2 and f_3, respectively.

The systems $(6.17)_r$, $r = 1, \ldots, N$, are all linear systems in α_r, β_r, γ_r. In particularly, the right side of the first equation of $(6.17)_r$ is independent of α_r, β_r, γ_r, which means that $\alpha_r(\tau)$ can be obtained easily from the first equation of $(6.17)_r$. The terms $P_{\alpha,r}(\tau), P_{\beta,r}(\tau), P_{\gamma,r}(\tau)$ in $(6.17)_r$ are polynomials in $\alpha_1, \beta_1, \gamma_1, ..., \alpha_{r-1}, \beta_{r-1}, \gamma_{r-1}$. Moreover, the coefficients of $P_{\alpha r}(\tau)$ depend on τ, $\beta_0(\tau)$,$\gamma_0(\tau)$, and $(X_0(0), Y_0(0), Z_0(0))$ and the derivatives of $(X_i(t)$, $Y_i(t)$, $Z_i(t))$ at $t = 0$ for $i = 0, 1, \ldots, r-1$, while the coefficients of $P_{\beta,r}(\tau), P_{\gamma,r}(\tau)$ depend on τ, $\beta_0(\tau)$,$\gamma_0(\tau)$, and $(X_0(0), Y_0(0), Z_0(0))$ and the derivatives of $(X_i(t), Y_i(t), Z_i(t))$ at $t = 0$ for $i = 0, 1, \ldots, r$. Hence for $r = 1, \ldots, N$, $P_{\alpha,r}(\tau)$, $P_{\beta,r}(\tau)$, $P_{\gamma,r}(\tau)$, are known recursively, if $(\alpha_{r-1}, \beta_{r-1}, \gamma_{r-1})$ can be obtained recursively for known $(X_i(t), Y_i(t), Z_i(t))$, $i = 0, 1, \ldots, N$. The constant terms of these polynomials are the sum of certain terms, each of which is a product of the factor of the difference between the derivatives of f_i at $(\xi_0$, $Y_0(0) + \beta_0(\tau)$, $Z_0(0) + \gamma_0(\tau), 0)$ and $(\xi_0$, $Y_0(0)$, $Z_0(0), 0)$ and some polynomial $q(\tau)$ in τ with degree $\leq r$.

6.2. Properties of the expansion of inner solutions. With the outer solution $(X(t, \epsilon), Y(t, \epsilon), Z(t, \epsilon))$ of Theorem 5.1 we can apply the change of variables (3.6a,b) and construct the boundary layer problem (3.7a,b). For the study of the inner problem (3.7a,b) we impose the following assumption on the solutions of $(6.17)_0$:

Assumption (IV):

(i) The initial value problem

$$\begin{aligned}
\frac{d\beta(\tau)}{d\tau} &= f_2(\xi_0, Y_0(0) + \beta, Z_0(0) + \gamma, 0), \\
0 &= f_3(\xi_0, Y_0(0) + \beta, Z_0(0) + \gamma, 0), \\
\beta(0) &= b, \quad \gamma(0) = c,
\end{aligned} \tag{6.18}$$

corresponding to $(6.17)_0$ has for

$$b = \eta_0 - Y_0(0) \text{ and } c = \zeta_0 - Z_0(0)$$

a solution $(\beta_0(\tau),\ \gamma_0(\tau))$ defined on $[0,\infty)$ such that
$\Omega_0(\tau) = (\xi_0, Y_0(0) + \beta_0(\tau), Z_0(0) + \gamma_0(\tau)) \in \mathcal{D}$ for all $\tau \geq 0$, and

$$\lim_{\tau\to\infty} (\beta_0(\tau), \gamma_0(\tau)) = (0,0).$$

(ii) The matrices

$$D_z f_3(\Omega_0(\tau),$$
$$C(\tau) = D_y f_2(\Omega_0(\tau)) - D_z f_2(\Omega_0(\tau))\,(D_z f_3(\Omega_0(\tau)))^{-1}\, D_y f_3(\Omega_0(\tau))$$

are nonsingular for $\tau \geq 0$.

Under the assumptions (I) - (IV) and with properly chosen ξ_r^*, $r = 1, \dots, N$, we shall prove in Proposition 6.1 that the system $(6.17)_r$ has a unique solution $(\alpha_r(\tau), \beta_r(\tau), \gamma_r(\tau))$ defined on $[0,\infty)$, $r = 1,\dots,N$, and that this solution decays to zero exponentially as $\tau \to \infty$.

The following result provides sufficient conditions under which Assumption (IV) can be derived from Assumptions (I) - (III):

Lemma 6.1: *If for the initial conditions $(\xi(\epsilon), \eta(\epsilon), \zeta(\epsilon))$ of (1.1) the point (ξ_0, η_0, ζ_0) is sufficiently close to the point $(\xi_0, Y_0(0), Z_0(0))$, then Assumption (IV) is a consequence of Assumptions (I) - (III).*

The principal part of Lemma 6.1 is a direct consequence of Lemma 6.3 below which in turn can be proved by means of the following result:

Lemma 6.2: *Let*
$$\frac{dx}{dt} = Ax + f(t,x)$$
where $A \in R^{n\times n}$ is a real (constant) matrix for which all eigenvalues have negative real parts. Let f be real continuous for $(x,t) \in B(0,\delta) \times R^+$, where $R^+ = \{t \mid t \geq 0\}$, and $\delta > 0$ is a small number, and let
$$f(t,x) = o(|x|) \quad \text{as } |x| \to 0$$
uniformly in t, $t \geq 0$. Then the identically zero solution is asymptotically stable.

For the proof see, e.g., [CoLe].

Lemma 6.3: *Let Assumptions (I) - (III) hold and consider the initial value problem (6.18) where the initial point (b,c) satisfies the constraint condition $f_3(\xi_0, Y_0(0) + b, Z_0(0) + c, 0) = 0$, and $(X_0(t), Y_0(t), Z_0(t))$ is the solution arising in Assumption (III). Then for sufficiently small $|b|, |c|$, the DAE (6.18) has exactly one solution $(\beta(\tau), \gamma(\tau))$ defined on $[0,\infty)$, for which $Y_0(0) + \beta(\tau) \in \mathcal{D}_y$, $Z_0(0) + \gamma(\tau) \in \mathcal{D}_z$ and*

$$\lim_{\tau\to\infty} (\beta(\tau), \gamma(\tau)) = (0,0).$$

Proof: By Assumption (I) the matrix $D_z f_3(\xi_0, Y_0(0), Z_0(0), 0)$ is nonsingular, $(Y_0(0), Z_0(0))$ is a point of the open set $\mathcal{D}_y \times \mathcal{D}_z$, and we have $f_3(\xi_0, Y_0(0), Z_0(0), 0) = 0$. By the implicit function theorem there are two positive scalars δ_β and δ_γ such that $Y_0(0) + B(0, \delta_\beta) \subset \mathcal{D}_y$, $Z_0(0) + B(0, \delta_\gamma) \subset \mathcal{D}_z$, and that for any $\beta \in B(0, \delta_\beta)$ there exists exactly one $\gamma = \phi(\beta) \in B(0, \delta_\gamma)$ for which $f_3(\xi_0, Y_0(0) + \beta, Z_0(0) + \gamma, 0) = 0$. Since the functions f_2 and f_3 were assumed to be sufficiently smooth, we may expand them at the point $(\xi_0, Y_0(0), Z_0(0), 0)$ and write (6.18) as

$$\begin{aligned} \frac{d\beta_0(\tau)}{d\tau} &= D_y f_2(\xi_0, Y_0(0), Z_0(0), 0)\beta \\ &\quad + D_y f_2(\xi_0, Y_0(0), Z_0(0), 0)\gamma + F(\beta, \gamma) \\ 0 &= D_y f_3(\xi_0, Y_0(0), Z_0(0), 0)\beta \\ &\quad + D_y f_3(\xi_0, Y_0(0), Z_0(0), 0)\gamma + G(\beta, \gamma) \end{aligned} \tag{6.19}$$

where $F(\beta, \gamma), G(\beta, \gamma) = O(|\beta|^2 + |\gamma|^2)$ as $\beta \to 0$ and $\gamma \to 0$. From the second equation of (6.19), it follows that

$$\begin{aligned} \gamma = &- (D_z f_3(\xi_0, Y_0(0), Z_0(0), 0))^{-1} D_y f_3(\xi_0, Y_0(0), Z_0(0), 0)\beta \\ &+ O(|\beta|^2 + |\gamma|^2) \end{aligned} \tag{6.20}$$

which, with $\phi(0) = 0$, implies that

$$\gamma = \phi(\beta) = O(\beta), \qquad \text{as } \beta \to 0. \tag{6.21}$$

Substituting (6.20) into (6.19) and using the asymptotic relation (6.21) we find that

$$\frac{d\beta}{d\tau} = B\beta + R(\beta). \tag{6.22}$$

Here

$$\begin{aligned} R(\beta) =& F(\beta, \gamma) - D_y f_2(\xi_0, Y_0(0), Z_0(0), 0)(D_z f_3(\xi_0, Y_0(0), Z_0(0), 0))^{-1} G(\beta, \gamma) \\ =& O(|\beta|^2 + |\gamma|^2) = O(|\beta)|^2), \qquad \text{as } \beta \to 0 \end{aligned}$$

and $B = B(0)$ is the matrix defined in Proposition 3.1 for which by Assumption (III) all eigenvalues have negative real parts. Thus by Lemma 6.2, $0 \in B(0, \delta_\beta)$ is an asymptotically stable solution of (6.22), namely, for any $0 < \epsilon < \delta_\beta$, there exists $\delta < \delta_\beta$ such that

$$|\beta(0)| < \delta \quad (\text{ or } |b| < \delta)$$

implies that the solution $\beta(\tau)$ of (6.22) satisfies

$$|\beta(\tau)| < \epsilon \quad \text{for all } \tau \geq 0$$

and

$$\lim_{\tau \to \infty} \beta(\tau) = 0.$$

Because $\beta(\tau)$ remains in the ball $B(0, \delta_\beta)$ for all $\tau \geq 0$, if follows from the continuity of ϕ and $\phi(0) = 0$, that $\gamma(\tau) = \phi(\beta(\tau)) \in B(0, \delta_\gamma)$ for all $\tau \geq 0$ and

$$\lim_{\tau \to \infty} \gamma(\tau) = 0.$$

This completes the proof of Lemma 6.3. ∎

Proof of Lemma 6.1: We apply Lemma 6.3 to $(6.17)_0$ to complete the proof of Lemma 6.1. Lemma 6.3 ensures that when (η_0, ζ_0) is sufficiently close to the point $(Y_0(0), Z_0(0))$ then $|\beta_0(0)|, |\gamma_0(0)|$ are sufficiently small and $(6.17)_0$ has exactly one solution $(\beta_0(\tau), \gamma_0(\tau))$ defined on $[0, \infty)$ which satisfies

$$\lim_{\tau \to \infty} (\beta_0(\tau), \gamma_0(\tau)) = (0, 0)$$

and

$$(Y_0(0) + \beta_0(\tau), Z_0(0) + \gamma_0(\tau)) \in \mathcal{D}_y \times \mathcal{D}_z, \quad \forall \tau \in [0, \infty).$$

It only remains to prove that the condition (ii) of Assumption (IV) holds. For this we can choose sufficiently small δ_β and δ_γ such that for all $(\beta, \gamma) \in B(0, \delta_\beta) \times B(0, \delta_\gamma)$ the matrices

$$\begin{aligned} &D_z f_3(\xi_0, Y_0(0) + \beta, Z_0(0) + \gamma, 0), \\ &D_y f_2 - D_z f_2 (D_z f_3)^{-1} D_y f_3(\xi_0, Y_0(0) + \beta, Z_0(0) + \gamma, 0), \end{aligned} \tag{6.23}$$

are nonsingular. This is guaranteed by the continuity of these two matrices and their nonsingularity at $(\beta, \gamma) = (0, 0)$. Then, for $\epsilon = \delta_\beta$ there exists a $\delta > 0$ such that

$$|\beta_0(0)| < \delta, \quad |\gamma_0(0)| < \delta,$$

implies that $(\beta_0(\tau), \gamma_0(\tau)) \in B(0, \delta) \times B(0, \delta)$ for all $\tau \geq 0$. Thus on the solutions $\beta_0(\tau)$, $\gamma_0(\tau)$ the matrices (6.23) are indeed nonsingular. This completes the proof of Lemma 6.1. ∎

Lemma 6.4: *Under Assumptions (I) - (III), if the solution $(\beta_0(\tau), \gamma_0(\tau))$ of $(6.17)_0$ satisfies the Assumption (IV), then there exists a positive σ such that*

$$\beta_0(\tau) = O(e^{-\sigma\tau}), \; \gamma_0(\tau) = O(e^{-\sigma\tau}) \tag{6.24}$$

as $\tau \to \infty$.

Proof: Since $\beta_0(\tau) \to 0$, $\gamma_0(\tau) \to 0$ as $\tau \to \infty$, for any given $\epsilon > 0$, we may choose a sufficiently large $\tau_0 > 0$ such that $\beta_0(\tau) \in B(0, \delta_\beta)$ and $\gamma_0(\tau) \in B(0, \delta_\gamma)$ for all $\tau \geq \tau_0$, and

$$|R(\beta_0(\tau))| < \epsilon$$

where the function R is defined in (6.22). The equation (6.22) can be written as the integral equation

$$\beta_0(\tau) = e^{-B\tau}\beta_0(\tau_0) + \int_{\tau_0}^{\tau} e^{-B(\tau-s)}R(\beta_0(s))ds. \tag{6.25}$$

Since the real parts of all eigenvalues of B are negative, it follows from Lemma 5.1 with $A(t) \equiv B$ and $\epsilon = 1$, that there exist positive numbers K and μ such that

$$\|e^{-B\tau}\| \leq Ke^{-\mu\tau}, \qquad \text{for all } \tau \geq 0. \tag{6.26}$$

Note that here we use the fact that the fundamental solution $\phi(t,s)$ of the equation

$$\frac{dy}{d\tau} = By$$

is $\phi(\tau,s) = e^{-(\tau-s)B}$, for $0 \leq s \leq \tau < \infty$. Hence (6.25) and (6.26) together show that

$$|\beta_0(\tau)| \leq e^{-\mu\tau}|\beta_0(\tau_0)| + \epsilon K \int_{\tau_0}^{\tau} e^{-\mu(\tau-s)}|\beta_0(s)|ds$$

or

$$e^{\mu\tau}|\beta_0(\tau)| \leq |\beta_0(\tau_0)| + \epsilon K \int_{\tau_0}^{\tau} e^{\mu s}|\beta_0(s)|ds$$

By Gronwall's inequality it follows that

$$|\beta_0(\tau)| \leq |\beta_0(\tau_0)|e^{-\epsilon K\tau_0}e^{-(\mu-\epsilon K)\tau} \tag{6.27}$$

and with $\epsilon = \frac{\mu}{2K}$ and $\sigma = \mu/2$ we obtain from (6.27) that

$$\beta_0(\tau) = O(e^{-\sigma\tau}) \qquad \text{as } \tau \to \infty$$

which, together with (6.21), completes the proof of Lemma 6.4. ∎

Lemma 6.4 shows that the first term of an inner solution of (3.7a,b) is negligible outside the boundary layer, which also is a property of any inner solution of (3.7a,b). Since $(\alpha_r(\tau), \beta_r(\tau), \gamma_r(\tau))$, $r = 0, 1, \ldots, N$, were derived as coefficients of an inner solution of (3.7a,b), we expect that all terms $(\alpha_r(\tau), \beta_r(\tau), \gamma_r(\tau))$, $r = 0, 1, \ldots, N$, possess this property, namely that , for $r = 1, \ldots, N$,

$$\lim_{\tau\to\infty}(\alpha_r(\tau), \beta_r(\tau), \gamma_r(\tau)) = (0, 0, 0).$$

This will be confirmed in the following Proposition 6.1 where also the existence and uniqueness of solutions $(\alpha_r(\tau), \beta_r(\tau), \gamma_r(\tau))$ of the systems $(6.17)_r$, $r = 1, \ldots, N$, is discussed.

By $(6.17)_r$ all coefficient functions $(\alpha_r, \beta_r, \gamma_r)$ satisfy a linear system of the form

$$\begin{aligned} \frac{d\alpha}{d\tau} &= P_1(\tau), \\ \frac{d\beta}{d\tau} &= A_{21}(\tau)\alpha + A_{22}(\tau)\beta + A_{23}(\tau)\gamma + P_2(\tau), \\ 0 &= A_{31}(\tau)\alpha + A_{32}(\tau)\beta + A_{33}(\tau)\gamma + P_3(\tau), \\ \alpha(0) = \alpha^*, &\quad \beta(0) = \beta^*, \quad \gamma(0) = \gamma^*, \end{aligned} \tag{6.28}$$

for which

(i) A_{ij}, P_j are of class C^1 on $[0, \infty)$ and $A_{33}(\tau)$ is nonsingular for all $\tau \geq 0$;

(ii) $A_{i,j}$ are bounded uniformly in $\tau \in [0, \infty)$ and the limit

$$\lim_{\tau \to \infty} \left(A_{22}(\tau) - A_{23}(\tau) A_{33}^{-1}(\tau) A_{32}(\tau) \right) = B_0$$

exists ;

(iii) the limiting matrix B_0 has all eigenvalues remaining strictly in the left half plane. And the initial point $(\alpha^*, \beta^*, \gamma^*)$ satisfies the compatibility condition

$$A_{31}(0)\alpha^* + A_{32}(0)\beta^* + A_{33}(0)\gamma^* + P_3(0) = 0.$$

For the system (6.28) we obtain the following result.

Lemma 6.5: *Under the conditions (i), (ii) and (iii) the system (6.28) has exactly one solution $(\alpha(\tau), \beta(\tau), \gamma(\tau))$ defined on the interval $[0, \infty)$. Moreover, the asymptotic relations*

$$P_i(\tau) = O(e^{-\sigma\tau}), \quad \text{as } \tau \to \infty, \quad i = 1, 2, 3$$

hold with some $\sigma > 0$. Then

$$\alpha^* = -\int_0^\infty P_1(s)ds < \infty \tag{6.29}$$

and with this α^* there exists a solution $(\alpha(\tau), \beta(\tau), \gamma(\tau))$ of (6.28) that satisfies

$$\alpha(\tau) = O(e^{-\sigma_1\tau}),\ \beta(\tau) = O(e^{-\sigma_1\tau}),\ \gamma(\tau) = O(e^{-\sigma_1\tau}), \quad \text{as } \tau \to \infty. \tag{6.30}$$

with some $0 < \sigma_1 \leq \sigma$.

Proof: From the first equation of (6.28) we obtain

$$\alpha(\tau) = \alpha^* + \int_0^\tau P_1(s)ds. \tag{6.31}$$

Hence since, by assumption (i), $A_{33}(\tau)$ is invertible, it follows from the third equation of (6.28) that

$$\gamma = -A_{33}^{-1}(\tau)\left(A_{31}(\tau)\alpha + A_{32}(\tau)\beta + P_3(\tau)\right). \tag{6.32}$$

Substituting (6.31) and (6.32) into the second equation of (6.28) we obtain

$$\frac{d\beta}{d\tau} = A(\tau)\beta + b(\tau) \tag{6.33a}$$

where

$$\begin{aligned} A(\tau) &= A_{22}(\tau) - A_{23}(\tau)A_{33}^{-1}(\tau)A_{32}(\tau), \\ b(\tau) &= P_2(\tau) - A_{23}(\tau)A_{33}^{-1}(\tau)P_3(\tau) + (A_{21}(\tau) - A_{23}(\tau)A_{33}^{-1}(\tau)A_{31}(\tau))\alpha(\tau). \end{aligned} \tag{6.33b}$$

Since (6.33a) is a linear ODE with the coefficients defined on $[0,\infty)$ it follows from the basic existence theorem for initial value problems of linear ODE's that with the initial condition $\beta(0) = \beta^*$ the equation (6.33a) has a unique solution $\beta = \beta(\tau)$ on $[0,\infty)$. Inserting $\beta = \beta(\tau)$ into (6.32) we obtain the component $\gamma = \gamma(\tau)$ of the solution for (6.28). This proves the first part of Lemma 6.5.

For the rest of this lemma, note that because of

$$P_1(\tau) = O(e^{-\sigma\tau}), \qquad \text{as } \tau \to \infty,$$

the integral

$$\int_0^\infty P_1(s)ds$$

exists. Then by choosing α^* as in (6.29) we find that the solution $\alpha(\tau)$ has the form

$$\alpha(\tau) = -\int_\tau^\infty P_1(s)ds$$

which implies that $\alpha(\tau)$ satisfies the asymptotic relation (6.30). Since A_{ij}, $i = 2,3, j = 1,2,3$, are bounded uniformly for $\tau \geq 0$, and $P_2(\tau)$, $P_3(\tau)$, $\alpha(\tau)$ satisfy the asymptotic relation

$$P_2(\tau) = O(e^{-\sigma\tau}),\ P_3(\tau) = O(e^{-\sigma\tau}),\ \alpha(\tau) = O(e^{-\sigma\tau}), \ \text{ as } \tau \to \infty$$

it follows that $b(\tau)$ in (6.33b) satisfies the same asymptotic relation, namely

$$b(\tau) = O(e^{-\sigma\tau}). \tag{6.34}$$

Since by conditions (ii) and (iii), $\lim_{\tau\to\infty} A(\tau) = B_0$ and all eigenvalues of B_0 remain strictly in the left half plane, there exist $\mu' > 0$ and $\tau_0 > 0$ such that for any eigenvalue $\lambda(A(\tau))$ of $A(\tau)$ the inequality

$$\Re(\lambda(A(\tau))) < -\mu' < 0, \ \ \forall\, \tau \geq \tau_0$$

holds. Let $\Phi(\tau, s)$ be a fundamental solution of the equation:

$$\begin{aligned} &\frac{d\Phi}{d\tau} = A(\tau)\Phi, \\ &\Phi(\tau,s)|_{\tau=s} = I, \qquad \tau_0 \leq s \leq \tau. \end{aligned}$$

From Lemma 5.1 with $\epsilon = 1$ we obtain the estimate

$$\Phi(t,s) = O(e^{-\mu'(t-s)/2})$$

Then for the solution $\beta(\tau)$ of (6.28), it follows that

$$\begin{aligned} \beta(\tau) &= \Phi(\tau,0)\beta^* + \int_0^\tau \Phi(\tau,s)b(s)ds \\ &= O(e^{-\mu'\tau/2}) + \int_0^\tau O(e^{-\mu'(\tau-s)/2})O(e^{-\sigma s})ds \qquad \text{as } \tau \to \infty \\ &= O(e^{-\mu'\tau/2}) + O(e^{-\sigma\tau}) \\ &= O(e^{-\sigma_1\tau}) \end{aligned}$$

where

$$\sigma_1 = \min\{\sigma, \mu'/2\}$$

Hence (6.32) implies that the asymptotic relation

$$\gamma(\tau) = O(e^{-\sigma_1\tau}) \qquad \text{as } \tau \to \infty$$

holds, which concludes the proof. ∎

Proposition 6.1: *Under the assumptions (I) - (IV), the system $(6.17)_r$ has a unique solution $(\alpha_r(\tau), \beta_r(\tau), \gamma_r(\tau))$, for $\tau \geq 0$ and $r = 1,\ldots,N$. Moreover, there exist coefficients ξ_r^*, $r = 1,\ldots,N$ of the expansion (4.1a) of $\xi^*(\epsilon)$ and positive numbers σ_r such that*

$$\left.\begin{aligned} \alpha_r(\tau) &= O(e^{-\sigma_r\tau}) \\ \beta_r(\tau) &= O(e^{-\sigma_r\tau}) \\ \gamma_r(\tau) &= O(e^{-\sigma_r\tau}) \end{aligned}\right\} \quad r = 1,\ldots,N \tag{6.35}$$

as $\tau \to \infty$.

Proof: The first part of this proposition is a direct consequence of Lemma 6.5 since, by Assumption (IV), $D_z f_3(\Omega_0(\tau))$ is nonsingular for all $\tau \geq 0$ and by the derivation of the system (6.17), $P_{\alpha,r}(\tau)$, $P_{\beta,r}(\tau)$ and $P_{\gamma,r}(\tau)$ are well defined for $\tau \geq 0$ provided only the previous terms $\alpha_i(\tau)$, $\beta_i(\tau)$, $\gamma_i(\tau)$, $i = 1,\ldots,r-1$, are available.

Hence it remains to show that with properly chosen coefficients ξ_r^*, $r = 1,\ldots,N$, the asymptotic relations (6.35) hold for all $0 \leq r \leq N$. This can be shown inductively by applying the methods used to prove the second part of Lemma 6.5. Indeed, Assumption (IV) ensures that the estimate (6.35) is valid for $r = 0$. Thus assume that there exist positive numbers $\sigma_1,\ldots,\sigma_k$ such that the asymptotic relations (6.35) hold for all $1 \leq r \leq k$. Note that the

coefficients $D_x f_i(\Omega_0(\tau))$, $D_y f_i(\Omega_0(\tau))$, $D_z f_i(\Omega_0(\tau))$, $i = 2, 3$, of the system $(6.17)_r$ are bounded uniformly in τ for $\tau \geq 0$. From the derivation of the system $(6.17)_r$ we know that $P_{\alpha,k+1}(\tau)$, $P_{\beta,k+1}(\tau)$ and $P_{\gamma,k+1}(\tau)$ are polynomials in α_1, β_1, γ_1,..., α_k, β_k, γ_k with coefficients depending on τ, $\beta_0(\tau)$, $\gamma_0(\tau)$ and $(X_0(0), Y_0(0), Z_0(0))$ and the higher derivatives of $(X_i(t), Y_i(t), Z_i(t))$ at $t = 0$. Moreover, for the constant terms of $P_{\alpha,k+1}(\tau)$, $P_{\beta,k+1}(\tau)$ and $P_{\gamma,k+1}(\tau)$, it follows from (6.16b) that they have the form

$$\sum_{j=1}^{k+1}(D^j f(\Omega_0(\tau)) - D^j f(\Gamma_0))p_{k+1j}(\tau) \tag{6.36}$$

where $p_{k+1,j}(\tau)$ is a polynomial with degree $\leq k+1$. Since

$$\Omega_0(\tau) - \Gamma_0 = \begin{pmatrix} 0 \\ \beta_0(\tau) \\ \gamma_0(\tau) \\ 0 \end{pmatrix},$$

the difference $f(\Omega_0(\tau)) - f(\Gamma_0)$ has the rate $O(|\beta_0(\tau)|+|\gamma_0(\tau)|)$ which, together with (6.36), implies that the constant terms of $P_{\alpha,k+1}(\tau)$, $P_{\beta,k+1}(\tau)$ and $P_{\gamma,k+1}(\tau)$ all have the rate

$$O((|\beta_0(\tau)| + |\gamma_0(\tau)|)\hat{p}(\tau)) \quad \text{as } \tau \to \infty \tag{6.37}$$

where $\hat{p}(\tau)$ is a polynomial of degree $\leq k+1$. Recall that for any given small number $\epsilon > 0$ and any polynomial $p_n(\tau)$, the limit

$$\lim_{\tau \to \infty} p_n(\tau)e^{-\epsilon\tau} = 0$$

exists. Moreover, by Lemma 6.4, $\beta_0(\tau), \gamma_0(\tau)$ have the rate $O(e^{-\sigma\tau})$ as $\tau \to \infty$. Thus, with $\sigma_0 = \sigma/2$, it follows from (6.37) that these constant terms have the rate

$$O((|\beta_0(\tau)| + |\gamma_0(\tau)|)\hat{p}(\tau)) = O(\hat{p}(\tau)e^{-\sigma\tau}) = O(\hat{p}(\tau)e^{-\sigma_0\tau})e^{-\sigma_0\tau} = O(e^{-\sigma_0\tau}) \quad \text{as } \tau \to \infty.$$

Hence there exists a positive number σ'_{k+1} such that the following asymptotic relations hold

$$\left.\begin{aligned} P_{\alpha,k+1}(\tau) &= O(e^{-\sigma'_{k+1}\tau}) \\ P_{\beta,k+1}(\tau) &= O(e^{-\sigma'_{k+1}\tau}) \\ P_{\gamma,k+1}(\tau) &= O(e^{-\sigma'_{k+1}\tau}) \end{aligned}\right\} \text{ as } \tau \to \infty$$

This implies that

$$\int_0^\infty P_{\alpha,k+1}(\tau)d\tau$$

exists. Therefore, by integrating the first equation of $(6.17)_{k+1}$ we see that

$$\alpha_{k+1}(\tau) = \xi_{k+1} - \xi^*_{k+1} + \int_0^\infty P_{\alpha k+1}(s)ds - \int_\tau^\infty P_{\alpha k+1}(s)ds.$$

Accordingly, we choose

$$\xi^*_{k+1} = \xi_{k+1} + \int_0^\infty P_{\alpha,k+1}(s)ds, \tag{6.38}$$

whence

$$\alpha_{k+1}(\tau) = -\int_\tau^\infty P_{\alpha,k+1}(s)ds = -\int_\tau^\infty O(e^{-\sigma'_{k+1}s})ds = O(e^{-\sigma'_{k+1}\tau}) \qquad \text{as } \tau \to \infty$$

and the asymptotic relation (6.35) holds for $\alpha_{k+1}(\tau)$.

Since $\beta_0(\tau)$, $\gamma_0(\tau) \to 0$ as $\tau \to \infty$, it follows that

$$\lim_{\tau\to\infty} \left\{ D_y f_2(\Omega_0(\tau)) - D_z f_2(\Omega_0(\tau))(D_z f_3(\Omega_0(\tau)))^{-1} D_y f_3(\Omega_0(\tau)) \right\} = B(0)$$

where by Assumption (III) all eigenvalues of $B(0)$ remain strictly in the left half plane. Hence, by applying Lemma 6.5 to $(6.17)_{k+1}$, with ξ^*_{k+1} chosen by (6.37), we obtain the existence of $\sigma_{k+1} > 0$ ($\leq \sigma'_{k+1}$) such that the other two components $\beta_{k+1}(\tau)$ and $\gamma_{k+1}(\tau)$ satisfy the asymptotic relations

$$\left.\begin{aligned} \beta_{k+1}(\tau) &= O(e^{-\sigma_{k+1}\tau}) \\ \gamma_{k+1}(\tau) &= O(e^{-\sigma_{k+1}\tau}) \end{aligned}\right\} \text{ as } \tau \to \infty.$$

Thus altogether we proved that the existence of ξ^*_{k+1} and of $\sigma_{k+1} > 0$ such that the asymptotic relation (6.35) holds for $r = k+1$ and, therefore, (6.35) is valid for all $0 \leq r \leq N$. ∎

6.3. The procedure for generating outer and inner solutions. The proofs of Lemma 6.5 and Proposition 6.1 describe the procedure for the generation of the sequences ξ^*_r and

$$(X_r(t), Y_r(t), Z_r(t)), \quad r = 1, \ldots, N \tag{6.39a}$$

$$(\alpha_r(\tau), \beta_r(\tau), \gamma_r(\tau)), \quad r = 1, \ldots, N \tag{6.39b}$$

which solve the systems $(4.2)_r$ and $(6.17)_r$, respectively, and satisfy the asymptotic relations (6.35) provided the first terms $(X_0(t),\ Y_0(t),\ Z_0(t))$ and $(0, \beta_0(\tau),\ \gamma_0(\tau))$ are available and satisfy Assumptions (I) - (IV). Suppose $(X_r(t),\ Y_r(t),\ Z_r(t))$ and $(\alpha_r(\tau),\ \beta_r(\tau),\ \gamma_r(\tau))$ are available for all $r \leq k$ and satisfy the requirements. Note that the polynomial $P_{\alpha,k+1}(\tau)$ in $\alpha_1, \ldots, \gamma_k$ depends only on the values of $(X_r(t),\ Y_r(t),\ Z_r(t))$ and their derivatives at $t = 0$ for

$r = 0, 1, \ldots, k$, and has the rate $P_{\alpha,k+1}(\tau) = O(e^{-\sigma_{k+1}\tau})$ as $\tau \to \infty$. Thus $P_{\alpha,k+1}(\tau)$ is known, and furthermore

$$\int_0^\infty P_{\alpha k+1}(\tau)d\tau$$

exists. Then with ξ^*_{k+1} specified by (6.38), with which we obtain the solution $\alpha_{k+1}(\tau)$ of the first equation of $(6.17)_{k+1}$. From the proof of Proposition 6.1, it follows that $\alpha_{k+1}(\tau)$ satisfies the asymptotic relation (6.35). With the initial condition $X_{k+1}(0) = \xi^*_{k+1}$ we can find the unique solution $(X_{k+1}(t),\ Y_{k+1}(t),\ Z_{k+1}(t))$ of the system $(4.2)_{k+1}$. Only at this moment, $P_{\beta k+1}(\tau)$ and $P_{\gamma k+1}(\tau)$ are fully determined because they depend not only on the previous solutions but also on the current one, $(X_{k+1}\ ,Y_{k+1},\ Z_{k+1})$. By inserting $\alpha_{k+1} = \alpha_{k+1}(\tau)$ into the last two equations of $(6.17)_{k+1}$, together with the initial conditions

$$\beta_{k+1}(0) = \eta_{k+1} - Y_{k+1}(0), \quad \gamma_{k+1}(0) = \zeta_{k+1} - Z_{k+1}(0)$$

we obtain the unique solution $(\beta_{k+1}(\tau), \gamma_{k+1}(\tau))$. In other words, our procedure has the following form.

Procedure (A): *Let $(X_0(t), Y_0(t), Z_0(t))$ and $(\beta_0(\tau), \gamma_0(\tau))$ be the solutions of the reduced problem (3.1) and of the problem $(6.17)_0$, respectively, which satisfy the Assumptions (I) - (IV). For $r = 1, \ldots, N$,*

(1) solve the ODE

$$\frac{d\alpha_r}{d\tau} = P_{\alpha,r}(\tau)$$
$$\alpha_r(0) = \xi_r - \xi^*_r$$

to obtain $\alpha_r(\tau)$ for

$$\xi^*_r = \xi_r + \int_0^\infty P_{\alpha,r}(s)ds;$$

(2) with the initial condition $X_r(0) = \xi^*_r$ solve $(4.2)_r$ to obtain the unique solution $(X_r(t)$, $Y_r(t)$, $Z_r(t))$;

(3) insert $\alpha_r(\tau)$ into the last two equations of $(6.17)_r$, to obtain the unique solution $(\beta_r(\tau)$, $\gamma_r(\tau))$.

Note that this procedure is indeed independent of the concepts of outer and inner solutions. But the sequences ξ^*_r and (6.39a,b) generated by this procedure can be used to construct outer and inner solutions. Next chapter shows that inner solutions exist and can be expanded with $(\alpha_r(\tau), \beta_r(\tau), \gamma_r(\tau))$ as coefficients.

7. Existence of inner solutions

7.1. Theorems of existences. The following theorem shows that the series

$$\sum_{r=0}^{N} \alpha_r(\tau)\epsilon^r, \quad \sum_{r=0}^{N} \beta_r(\tau)\epsilon^r, \quad \sum_{r=0}^{N} \gamma_r(\tau)\epsilon^r$$

generated by Procedure (A) can be used to approximate an inner solution ($\alpha(\tau,\epsilon)$, $\beta(\tau,\epsilon)$, $\gamma(\tau,\epsilon)$).

Theorem 7.1: *Under Assumptions (I) - (IV) suppose that the sequences* ξ_r^*, $(X_r(t), Y_r(t), Z_r(t))$ *and* $(\alpha_r(\tau), \beta_r(\tau), \gamma_r(\tau))$, $r = 0, 1, \dots, N$, *are generated by Procedure (A). Let* $(X(t,\epsilon), Y(t,\epsilon), Z(t,\epsilon)) \in \mathcal{D}$ *for* $t \in [0,T]$ *be the outer solution guaranteed to exist by Theorem 5.1 which satisfies the initial condition*

$$X(0,\epsilon) = \xi^*(\epsilon) \sim \xi_0 + \sum_{i=1}^{\infty} \xi_i^* \epsilon^i,$$

and the asymptotic relations (5.1). Then there exists a unique solution $(\alpha(\tau,\epsilon), \beta(\tau,\epsilon), \gamma(\tau,\epsilon))$ *for the inner problem (3.7) on the interval* $0 \le \tau \le T/\epsilon$, *for all* $0 < \epsilon \le \epsilon_1$, *where* ϵ_1 $(\le \epsilon_0)$ *is a sufficiently small number such that*

$$\begin{aligned} X(t,\epsilon) + \alpha(t/\epsilon,\epsilon) &\in \mathcal{D}_x, \\ Y(t,\epsilon) + \beta(t/\epsilon,\epsilon) &\in \mathcal{D}_y, \\ Z(t,\epsilon) + \gamma(t/\epsilon,\epsilon) &\in \mathcal{D}_z, \end{aligned}$$

for all $t \in [0,T]$ *and* $0 < \epsilon \le \epsilon_1$, *and*

$$\begin{aligned} \alpha(\tau,\epsilon) - \sum_{r=0}^{N} \alpha_r(\tau)\epsilon^r &= O(\epsilon^{N+1}), \\ \beta(\tau,\epsilon) - \sum_{r=0}^{N} \beta_r(\tau)\epsilon^r &= O(\epsilon^{N+1}), \\ \gamma(\tau,\epsilon) - \sum_{r=0}^{N} \gamma_r(\tau)\epsilon^r &= O(\epsilon^{N+1}), \end{aligned} \tag{7.1}$$

uniformly for $0 \le \tau \le T/\epsilon$ *as* $\epsilon \to 0$.

The proof of this theorem will be given after next theorem. From Theorem 5.1 and Theorem 7.1 follows the main Theorem of this paper.

Theorem 7.2: *Let Assumptions (I) - (IV) hold. Then there exists an* $\epsilon_2 > 0$, $0 < \epsilon_2 \le \epsilon_1$, *where* ϵ_1 *is defined in Theorem 7.1, such that for all* $0 < \epsilon \le \epsilon_2$ *the singularly perturbed DAE (1.1a,b) has a unique solution* $x = x(t,\epsilon)$, $y = y(t,\epsilon)$, $z = z(t,\epsilon)$ *in* $\mathcal{D}$ *on the interval* $0 \le t \le T$ *. Moreover, for any natural number* N, *there exist an outer solution* $(X(t,\epsilon), Y(t,\epsilon), Z(t,\epsilon))$ $\in \mathcal{D}$, *and an inner solution* $(\alpha(\tau,\epsilon), \beta(\tau,\epsilon), \gamma(\tau,\epsilon))$ *such that*

$$x(t,\epsilon) = X(t,\epsilon) + \alpha(t/\epsilon,\epsilon),\ y(t,\epsilon) = Y(t,\epsilon) + \beta(t/\epsilon,\epsilon),\ z(t,\epsilon) = Z(t,\epsilon) + \gamma(t/\epsilon,\epsilon)$$

for $0 \le t \le T$ and $0 < \epsilon \le \epsilon_0$. *Moreover, the outer solution satisfies*

$$X(t,\epsilon) - \sum_{r=0}^{N} X_r(\tau)\epsilon^r = O(\epsilon^{N+1})$$
$$Y(t,\epsilon) - \sum_{r=0}^{N} Y_r(\tau)\epsilon^r = O(\epsilon^{N+1})$$
$$Z(t,\epsilon) - \sum_{r=0}^{N} Z_r(\tau)\epsilon^r = O(\epsilon^{N+1})$$

uniformly for $0 \le t \le T$, *where* $(X_r(t), Y_r(t), Z_r(t))$ *are determined by the system* $(4.2)_r$, $r = 0, 1, \ldots, N$, *and the inner solution satisfies*

$$\alpha(t/\epsilon,\epsilon) - \sum_{r=0}^{N} \alpha_r(t/\epsilon)\epsilon^r = O(\epsilon^{N+1})$$
$$\beta(t/\epsilon,\epsilon) - \sum_{r=0}^{N} \beta_r(t/\epsilon)\epsilon^r = O(\epsilon^{N+1})$$
$$\gamma(t/\epsilon,\epsilon) - \sum_{r=0}^{N} \gamma_r(t/\epsilon)\epsilon^r = O(\epsilon^{N+1})$$

uniformly for $0 \le t \le T$, *where* $(\alpha_r(\tau), \beta_r(\tau), \gamma_r(\tau))$ *are determined by* $(6.17)_r$, *and satisfy*

$$\alpha_r(\tau) = O(e^{-\sigma_r\tau}),\ \beta_r(\tau) = O(e^{-\sigma_r\tau}),\ \gamma_r(\tau) = O(e^{-\sigma_r\tau}), \quad , \ r = 0, 1, \ldots, N, \quad \text{as } \tau \to \infty$$

for some $\sigma_r > 0$.

7.2. Proof of Theorem 7.1. In analogy to the proof of Theorem 5.1, the first step here introduces a change of variables into the inner system (3.7) in order to change it to a technically simpler form. Let $u \in R^m, v \in R^n, w \in R^k$ be defined by

$$\begin{aligned}\alpha &= u + \sum_{r=0}^{N} \alpha_r(\tau)\epsilon^r,\\ \beta &= v + \sum_{r=0}^{N} \beta_r(\tau)\epsilon^r + A_1(\tau)u,\\ \gamma &= w + \sum_{r=0}^{N} \gamma_r(\tau)\epsilon^r + A_2(\tau)u + B_1(\tau)v,\end{aligned} \tag{7.2}$$

where $A_1(\tau)$, $A_2(\tau)$, and $B_1(\tau)$ will be determined such that after the change of variables (7.2) the system (3.7) will simplify. At first we have to determine again a proper domain for the new

variables (u, v, w) such that the transformation (7.2) makes sense for the inner problem (3.7). For this recall that the inner system (3.7) was obtained by introducing the change of variables (3.6a/b) into (1.1) where $(X(t,\epsilon), Y(t,\epsilon), Z(t,\epsilon))$ is an outer solution. Thus the change of variables (7.2) in the inner system (3.7) is equivalent with the following change of variables in the original DAE (1.1):

$$
\begin{aligned}
x &= X(t,\epsilon) + u + \sum_{r=0}^{N} \alpha_r(t/\epsilon)\epsilon^r, \\
y &= Y(t,\epsilon) + v + \sum_{r=0}^{N} \beta_r(t/\epsilon)\epsilon^r + A_1(t/\epsilon)u, \\
z &= Z(t,\epsilon) + w + \sum_{r=0}^{N} \gamma_r(t/\epsilon)\epsilon^r + A_2(t/\epsilon)u + B_1(t/\epsilon)v.
\end{aligned}
\tag{7.3}
$$

It will be shown that there is a small neighborhood O_δ of $(0,0,0)$ such that if (u, v, w) remains in O_δ, then $(x, y, z) \in \mathcal{D}$ where (x, y, z) is defined by (7.3). For this, we show first that under Assumptions (I) - (IV) there exists a sufficiently small $0 < \epsilon_3$ ($\leq \epsilon_2$) such that $(Y_0(t)+\beta_0(t/\epsilon), Z_0(t) + \gamma_0(t/\epsilon)) \in \mathcal{D}_y \times \mathcal{D}_z$ for all $t \in [0, T]$, $0 < \epsilon \leq \epsilon_3$. Indeed, since, by Assumption (IV), $(Y_0(0) + \beta_0(\tau), Z_0(0) + \gamma_0(\tau)) \in \mathcal{D}_y \times \mathcal{D}_z$ for all $\tau \geq 0$ and $\mathcal{D}_y$, $\mathcal{D}_z$ are open sets, there exists $t^* \in (0, T)$ such that

$$
(Y_0(t) + \beta_0(t/\epsilon), Z_0(t) + \gamma_0(t/\epsilon)) \in \mathcal{D}_y \times \mathcal{D}_z, \quad \text{for all } t \in [0, t^*], 0 < \epsilon \leq \epsilon_3. \tag{7.4}
$$

On the other hand, since $(Y_0(t), Z_0(t)) \in \mathcal{D}_y \times \mathcal{D}_z$ for all $t \in [0, T]$ and $[0, T]$ is a closed set, there exists $r > 0$, independent of t, such that

$$
B(Y_0(t), r) \times B(Z_0(t), r) \subset \mathcal{D}_y \times \mathcal{D}_z, \quad \forall t \in [0, T], \tag{7.5}
$$

From $\lim_{\tau\to\infty}(\beta_0(\tau), \gamma_0(\tau)) = (0,0)$, we find that there exists $0 < \epsilon_3'$ ($\leq \epsilon_3$) such that

$$
|\beta_0(t/\epsilon)| < r, \quad |\gamma_0(t/\epsilon)| < r, \quad \forall t \in [t^*, T], 0 < \epsilon \leq \epsilon_3', \tag{7.6}
$$

Hence (7.5) and (7.6) show that

$$
(Y_0(t) + \beta_0(t/\epsilon), Z_0(t) + \gamma_0(t/\epsilon)) \in \mathcal{D}_y \times \mathcal{D}_z, \quad \text{for all } t \in [t^*, T], 0 < \epsilon \leq \epsilon_3', \tag{7.7}
$$

which, together with (7.4), means that

$$
(Y_0(t) + \beta_0(t/\epsilon), Z_0(t) + \gamma_0(t/\epsilon)) \in \mathcal{D}_y \times \mathcal{D}_z, \quad \text{for all } t \in [0, T], 0 < \epsilon \leq \epsilon_3'. \tag{7.8}
$$

By Theorem 5.1, the estimates

$$\left.\begin{aligned} X(t,\epsilon) &= X_0(t) + O(\epsilon) \\ Y(t,\epsilon) &= Y_0(t) + O(\epsilon) \\ Z(t,\epsilon) &= Z_0(t) + O(\epsilon) \end{aligned}\right\} \quad \text{as } \epsilon \to 0, \tag{7.9}$$

hold uniformly in $t \in [0,T]$. Therefore, (7.8), together with (7.9), implies that there exists $\epsilon_4 > 0$ ($\leq \epsilon_3'$) such that

$$(X(t,\epsilon), Y(t,\epsilon) + \beta_0(t/\epsilon), Z(t,\epsilon) + \gamma_0(t/\epsilon)) \in \mathcal{D}_x \times \mathcal{D}_y \times \mathcal{D}_z, \quad \forall t \in [0,t^*], 0 < \epsilon \leq \epsilon_4. \tag{7.10}$$

Note that $(\alpha_i(t/\epsilon), \beta_i(t/\epsilon), \gamma_i(t/\epsilon))$, $i = 1,\ldots,N$, are uniformly bounded for all $t \in [0,T]$ and $0 < \epsilon \leq \epsilon_4$. Moreover, suppose that $A_1(\tau), A_2(\tau)$ and $B_1(\tau)$ are uniformly bounded for $0 \leq \tau \leq T/\epsilon$. Then there exist $\epsilon_5 > 0$ ($\leq \epsilon_4$) and $\delta > 0$ such that for any $(u,v,w) \in B(0,\delta) \times B(0,\delta) \times B(0,\delta)$, the point (x,y,z) defined by the transformation (7.3) is contained in $\mathcal{D}$ for all $t \in [0,T]$ and $0 < \epsilon \leq \epsilon_5$. We write $\mathcal{D}_u = B(0,\delta) \subset R^m$, $\mathcal{D}_v = B(0,\delta) \subset R^n$, $\mathcal{D}_w = B(0,\delta) \subset R^k$, and $\hat{\mathcal{D}} = \mathcal{D}_u \times \mathcal{D}_v \times \mathcal{D}_w$.

The change of variable (7.2) is now introduced in (3.7). For abbreviation, the notations $\Omega_i(\tau)$, $\Gamma_i(\tau)$ defined in (6.10) and (6.11) will be used again. Furthermore, set

$$\mathcal{A} = \begin{pmatrix} I & 0 & 0 & 0 \\ A_1(\tau) & I & 0 & 0 \\ A_2(\tau) & B_1(\tau) & I & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{U} = \begin{pmatrix} u \\ v \\ w \\ 0 \end{pmatrix}.$$

Then

$$\Omega = \begin{pmatrix} x \\ y \\ z \\ \epsilon \end{pmatrix} = \Omega_0(\tau) + \sum_{i=1}^{N} \Omega_i(\tau)\epsilon^i + \mathcal{A}(\tau)\mathcal{U} + R_n(\tau,\epsilon) \tag{7.11}$$

and

$$\Gamma = \begin{pmatrix} X(\epsilon\tau,\epsilon) \\ Y(\epsilon\tau,\epsilon) \\ Z(\epsilon\tau,\epsilon) \\ \epsilon \end{pmatrix} = \Gamma_0 + \sum_{i=1}^{N} \Gamma_i(\tau)\epsilon^i + R_n(\tau,\epsilon) \tag{7.12}$$

where $R_n(\tau,\epsilon) = O(\epsilon^{N+1})$ is independent of (u,v,w). Thus, by expanding $f(\Omega)$ and $f(\Gamma)$ at $\Omega_0(\tau)$ and Γ_0, respectively, we obtain

$$\begin{aligned} &f(\Omega) - f(\Gamma) \\ &= f(\Omega_0(\tau)) + \sum_{j=1}^{N} \frac{D^j f(\Omega_0(\tau))}{j!} \left(\sum_{k=1}^{N} \Omega_k(\tau)\epsilon^k + \mathcal{A}(\tau)\mathcal{U} \right)^j + R(\tau,u,v,w,\epsilon) \\ &\quad - f(\Gamma_0) - \sum_{j=1}^{N} \frac{D^j f(\Gamma_0)}{j!} \left(\sum_{k=1}^{N} \Gamma_k(\tau)\epsilon^k \right)^j - H(\tau,\epsilon). \end{aligned} \tag{7.13}$$

As a remainder term, $R(\tau, u, v, w, \epsilon)$ satisfies

(i) $R(t/\epsilon, 0, 0, 0, \epsilon) = O(\epsilon^N)$, as $\epsilon \to 0$ (7.14a);

(ii)

$$\left.\begin{aligned} D_u R(t/\epsilon, u, v, w, \epsilon) &= O(\epsilon + |u| + |v| + |w|) \\ D_v R(t/\epsilon, u, v, w, \epsilon) &= O(\epsilon + |u| + |v| + |w|) \\ D_w R(t/\epsilon, u, v, w, \epsilon) &= O(\epsilon + |u| + |v| + |w|) \end{aligned}\right\} \quad \text{as } \epsilon, |u|, |v|, |w| \to 0 \tag{7.14b}$$

uniformly for $0 \le t \le T$.

Moreover the remainder term $H(\tau, \epsilon) = O(\epsilon^{N+1})$ satisfies (7.14a,b) as well. Obviously, the conditions (7.14a,b) are exactly the same as **Condition (N)**. It follows from (7.13) that

$$\begin{aligned} &f(\Omega) - f(\Gamma) \\ &= f(\Omega_0(\tau)) - f(\Gamma_0) + \sum_{r=1}^{N} c_r \epsilon^r + Df(\Omega_0(\tau))\mathcal{A}(\tau)\mathcal{U} + G(\tau, u, v, w, \epsilon) \end{aligned} \tag{7.15}$$

where c_r is as defined in (6.15) and

$$\begin{aligned} G(\tau, u, v, w, \epsilon) &= R(\tau, u, v, w, \epsilon) + O(\epsilon^{N+1}) - H(\tau, \epsilon) \\ &+ \sum_{j=2}^{N} \frac{D^j f(\Omega_0(\tau))}{j!} [\left(\sum_{k=1}^{N} \Omega_k(\tau)\epsilon^k + \mathcal{A}(\tau)\mathcal{U}\right)^j - \left(\sum_{k=1}^{N} \Omega_k(\tau)\epsilon^k\right)^j]. \end{aligned}$$

Under our assumption that $\mathcal{A}(\tau)$ is uniformly bounded on $0 \le \tau \le T/\epsilon$, we see that

$$\sum_{j=2}^{N} \frac{D^j f(\Omega_0(\tau))}{j!} [\left(\sum_{k=1}^{N} \Omega_k(\tau)\epsilon^k + \mathcal{A}(\tau)\mathcal{U}\right)^j - \left(\sum_{k=1}^{N} \Omega_k(\tau)\epsilon^k\right)^j]$$

satisfies (7.14a,b). Since the **Condition (N)** also holds for $R(\tau, u, v, w, \epsilon)$, $O(\epsilon^{N+1})$, $H(\tau, \epsilon)$, this implies that $G(\tau, u, v, w, \epsilon)$ satisfies **Condition (N)** as well.

Substituting (7.2) into the first equation of (3.7a) we obtain

$$\begin{aligned} &\frac{du}{d\tau} + \sum_{r=0}^{N} \frac{d\alpha_r}{d\tau}\epsilon^r = \epsilon(f_1(\Omega) - f_1(\Gamma)) \\ &= \epsilon[f_1(\Omega_0(\tau)) - f_1(\Gamma_0) + \sum_{r=1}^{N} c_r \epsilon^r + Df_1(\Omega_0(\tau))\mathcal{A}\mathcal{U}] + G_1(\tau, u, v, w, \epsilon). \end{aligned} \tag{7.16}$$

Since $\alpha_r(\tau)$ satisfies the first equation of $(6.17)_r$, it follows from (7.16) that

$$\frac{du}{d\tau} = \epsilon Df_1(\Omega_0(\tau))\mathcal{A}(\tau)\mathcal{U} + G_1(\tau, u, v, w, \epsilon) \tag{7.17}$$

where G_1 satisfies (7.14a,b). Similarly, substituting (7.2) into the second equation in (3.7a), and utilizing the fact that $\beta_r(\tau)$ is the solution of $(6.17)_r$, $r = 1, \ldots, N$, we see that

$$\begin{aligned} &\frac{dv}{d\tau} + A_1'(\tau)u + A_1(\tau)\frac{du}{d\tau} \\ &= Df_2(\Omega_0(\tau))\mathcal{A}(\tau)\mathcal{U} + G^{(2)}(\epsilon, u, v, w, \tau) \end{aligned} \tag{7.18}$$

where $G^{(2)}$is defined by (7.15) with f replaced by f_2. Obviously, the function on the right side of (7.18) satisfies (7.14a,b). Thus

$$\begin{aligned} &G_2(\epsilon, u, v, w, \tau) \\ &= -\epsilon A_1(\tau)Df_1(\Omega_0(\tau))\mathcal{A}(\tau)\mathcal{U} - A_1(\tau)G_1(\tau, u, v, w, \epsilon) + G^{(2)}(\epsilon, u, v, w, \tau) \end{aligned} \tag{7.19}$$

satisfies (7.14a,b) as well, and it follows from (7.18) that

$$\frac{dv}{d\tau} = Df_2(\Omega_0(\tau))\mathcal{A}(\tau)\mathcal{U} - A_1'(\tau)u + G_2(\epsilon, u, v, w, \tau). \tag{7.20}$$

Finally, substituting (7.2) into the third equation of (3.7a), we find that

$$Df_3(\Omega_0(\tau))\mathcal{A}(\tau)\mathcal{U} + G_3(\tau, u, v, w, \epsilon) = 0 \tag{7.21}$$

where G_3 is defined by (7.15) with f replaced by f_3.

Note that from the derivations of the equations (7.17), (7.20) and (7.21), we see that the functions G_i, $i = 1, 2, 3$, are well defined on $\hat{\mathcal{D}} \times (0, \epsilon_5]$. Now we show that there exist matrices $A_1(\tau)$, $A_2(\tau)$ and $B_1(\tau)$ which are uniformly bounded on $0 \le \tau \le T/\epsilon$ and for which the equations (7.17), (7.20) and (7.21) simplify considerably. For this, we write

$$\begin{aligned} &Df_2(\Omega_0(\tau))\mathcal{A}(\tau)\mathcal{U} - A_1'(\tau)u \\ &= [D_x f_2(\Omega_0(\tau)) + D_y f_2(\Omega_0(\tau))A_1(\tau) + D_z f_2(\Omega_0(\tau)A_2(\tau) - A_1'(\tau)]u \\ &\quad + [D_y f_2(\Omega_0(\tau)) + D_z f_2(\Omega_0(\tau))B_1(\tau)]v + D_z f_2(\Omega_0(\tau))w \end{aligned}$$

and

$$\begin{aligned} &Df_3(\Omega_0(\tau))\mathcal{A}(\tau)\mathcal{U} \\ &= [D_x f_3(\Omega_0(\tau)) + D_y f_3(\Omega_0(\tau))A_1(\tau) + D_z f_3(\Omega_0(\tau)A_2(\tau)]u \\ &\quad + [D_y f_2(\Omega_0(\tau)) + D_z f_3(\Omega_0(\tau))B_1(\tau)]v + D_z f_3(\Omega_0(\tau))w, \end{aligned} \tag{7.22}$$

and use the equations

$$\begin{aligned} &D_x f_2(\Omega_0(\tau)) + D_y f_2(\Omega_0(\tau))A_1(\tau) + D_z f_2(\Omega_0(\tau))A_2(\tau) - A_1'(\tau) = 0, \\ &D_x f_3(\Omega_0(\tau)) + D_y f_3(\Omega_0(\tau))A_1(\tau) + D_z f_3(\Omega_0(\tau))A_2(\tau) = 0, \\ &D_y f_3(\Omega_0(\tau)) + D_z f_3(\Omega_0(\tau))B_1(\tau) = 0, \end{aligned} \tag{7.23}$$

to determine $A_1(\tau), A_2(\tau)$ and $B_1(\tau)$. In order to show that the solutions $A_1(\tau), A_2(\tau)$ and $B_1(\tau)$ of (7.23) are uniformly bounded for $0 \le \tau \le T/\epsilon$, note first that the third equation of (7.23) gives

$$B_1(\tau) = -\left(D_z f_3(\Omega_0(\tau))\right)^{-1} D_y f_3(\Omega_0(\tau))$$

which is obviously bounded for $0 \le \tau \le T/\epsilon$. Indeed, there is a constant C, independent of τ and ϵ, such that $\|B_1(\tau)\| \le C$ for all $\tau \ge 0$. The second equation of (7.23) now provides that

$$A_2(\tau) = -\left(D_z f_3(\Omega_0(\tau))\right)^{-1} \left(D_x f_3(\Omega_0(\tau)) + D_y f_3(\Omega_0(\tau)) A_1(\tau)\right). \tag{7.24}$$

By substituting this into the first equation of (7.23) we obtain that

$$\frac{d}{d\tau} A_1(\tau) = M(\tau) A_1(\tau) + C(\tau) \tag{7.25}$$

where

$$\begin{aligned} M(\tau) &= D_y f_2(\Omega_0(\tau)) - D_z f_2(\Omega_0(\tau)) \left(D_z f_3(\Omega_0(\tau))\right) D_y f_3(\Omega_0(\tau)), \\ C(\tau) &= D_x f_2(\Omega_0(\tau)) - D_z f_2(\Omega_0(\tau)) \left(D_z f_3(\Omega_0(\tau))\right) D_x f_3(\Omega_0(\tau)). \end{aligned} \tag{7.26}$$

Because of

$$\lim_{\tau \to \infty} M(\tau) = B(0)$$

and $\Re(\lambda(B(0))) < 0$ for all eigenvalues $\lambda(B(0))$ of $B(0)$, there exists a $\tau_0 > 0$ such that all eigenvalues of $M(\tau)$ remain strictly in left half plane for $\tau \ge \tau_0$. Thus if $A_1(\tau)$ is chosen as the solution of (7.25) that satisfies the initial condition $A_1(0) = I$, then Lemma 5.1 ensures that $A_1(\tau)$ is uniformly bounded in τ, $\tau \ge 0$. With this $A_1(\tau)$ it follows directly from (7.24) that $A_2(\tau)$ is uniformly bounded in τ as well.

Note that

$$\begin{aligned} u(0,\epsilon) &= \alpha(0,\epsilon) - \sum_{r=0}^{N} \alpha_r(0)\epsilon^r \\ &= \xi(\epsilon) - \xi^*(\epsilon) - \sum_{r=0}^{N} (\xi_r - \xi_r^*)\epsilon^r \\ &= \tilde{\theta}_1(\epsilon) = O(\epsilon^{N+1}), \end{aligned} \tag{7.27a}$$

$$\begin{aligned} v(0,\epsilon) &= \beta(0,\epsilon) - \sum_{r=0}^{N} \beta_r(0)\epsilon^r - A_1(0)u(0,\epsilon) \\ &= \tilde{\theta}_2(\epsilon) = O(\epsilon^{N+1}), \end{aligned} \tag{7.27b}$$

$$\begin{aligned} w(0,\epsilon) &= \gamma(0,\epsilon) - \sum_{r=0}^{N} \gamma_r(0)\epsilon^r - A_2(0)u(0,\epsilon) - B_1(0)v(0,\epsilon) \\ &= \tilde{\theta}_3(\epsilon) = O(\epsilon^{N+1}). \end{aligned} \tag{7.27c}$$

Therefore, with these specially chosen $A_1(\tau)$, $A_2(\tau)$ and $B_1(\tau)$, it follows from (7.17), (7.20), (7.21) and (7.27a,b,c) that (3.7) becomes

$$\begin{aligned}\frac{du}{d\tau} &= \epsilon(C_1(\tau)u + L_1(\tau)v + E_1(\tau)w) + \epsilon G_1(\epsilon, u, v, w, \tau)\\ \frac{dv}{d\tau} &= M(\tau)v + E_2(\tau)w + G_2(\epsilon, u, v, w, \tau)\\ 0 &= E_3(\tau)w + G_3(\epsilon, u, v, w, \tau)\\ u(0,\epsilon) &= \tilde{\theta}_1(\epsilon) = O(\epsilon^{N+1}),\\ v(0,\epsilon) &= \tilde{\theta}_2(\epsilon) = O(\epsilon^{N+1}),\\ w(0,\epsilon) &= \tilde{\theta}_3(\epsilon) = O(\epsilon^{N+1}).\end{aligned} \tag{7.28}$$

This is equivalent to the following integral equation

$$\begin{aligned}u(\tau,\epsilon) &= \Phi(\tau,0,\epsilon)\tilde{\theta}_1(\epsilon) + \int_0^\tau \Phi(\tau,s,\epsilon)\epsilon(L_1(s)v(s,\epsilon) + E_1(s)w(s,\epsilon)\\ &\quad + G_1(\epsilon, u(s,\epsilon), v(s,\epsilon), w(s,\epsilon), s))ds,\\ v(\tau,\epsilon) &= \Psi(\tau)\tilde{\theta}_2(\epsilon) + \Psi(\tau)\int_0^\tau \Psi^{-1}(s)(E_2(s)w(s,\epsilon)\\ &\quad + G_2(\epsilon, u(s,\epsilon), v(s,\epsilon), w(s,\epsilon), s))ds,\\ w(\tau,\epsilon) &= -E_3^{-1}(\tau)G_3(\epsilon, u(\tau,\epsilon), v(\tau,\epsilon), w(\tau,\epsilon), \tau),\end{aligned} \tag{7.29}$$

where

$$\begin{aligned}C_1(\tau) &= D_x f_1(\Omega_0(\tau)) + D_y f_1(\Omega_0(\tau))A_1(\tau) + D_z f_1(\Omega_0(\tau))A_2(\tau),\\ L_1(\tau) &= D_y f_1(\Omega_0(\tau)) + D_z f_1(\Omega_0(\tau))B_1(\tau),\\ E_1(\tau) &= D_z f_1(\Omega_0(\tau)),\quad E_2(\tau) = D_z f_2(\Omega_0(\tau)),\quad E_3(\tau) = D_z f_3(\Omega_0(\tau)),\end{aligned}$$

and $\Phi(\tau, s, \epsilon)$ satisfies

$$\begin{aligned}\frac{d\Phi(\tau,s,\epsilon)}{d\tau} &= \epsilon C_1(\tau)\Phi(\tau,s,\epsilon)\\ \Phi(\tau,s,\epsilon)|_{\tau=s} &= I\end{aligned} \tag{7.30}$$

while $\Psi(\tau)$ is a solution of following system

$$\begin{aligned}\frac{d\Psi}{d\tau} &= M(\tau)\Psi\\ \Psi(0) &= I.\end{aligned} \tag{7.31}$$

It turns out that $\Phi(\tau, s, \epsilon)$, $\Psi(\tau)$ are uniformly bounded for $0 \leq s \leq \tau \leq T/\epsilon$ and $0 < \epsilon \leq \epsilon_0$. Indeed, because $C_1(\tau)$ is uniformly bounded for all $\tau \geq 0$, there exists a constant $C > 0$ such that $|C_1(\tau)| \leq C$ for all $\tau \geq 0$. Thus from

$$\Phi(\tau,s,\epsilon) = I + \epsilon\int_s^\tau C_1(\lambda)\Phi(\lambda,s,\epsilon)d\lambda$$

it follows that

$$\|\Phi(\tau, s, \epsilon)\| \leq 1 + \epsilon C \int_s^\tau \|\Phi(\lambda, s, \epsilon)\| d\lambda$$

which by Gronwall's inequality implies that

$$\|\Phi(\tau, s, \epsilon)\| \leq e^{\epsilon C(\tau - s)} \leq e^{CT} \tag{7.32}$$

for all $0 \leq s \leq \tau \leq T/\epsilon$. Hence $\Phi(\tau, s, \epsilon)$ is uniformly bounded for $0 \leq s \leq \tau \leq T/\epsilon$. For the matrix $M(\tau)$, there exists a positive number μ such that

$$\Re(\lambda(M(\tau))) < -\mu, \quad \tau \geq \tau_0$$

holds for all eigenvalues of $M(\tau)$. Hence by Lemma 5.1 (with $\epsilon = 1$) it follows that

$$\|\Psi(\tau)(\Psi(s))^{-1}\| \leq K e^{-\mu(\tau - s)/2} \tag{7.33}$$

holds for $0 \leq s \leq \tau$, where K is a constant which is independent of s, τ.

Now we show that (7.29) has a unique solution (u, v, w) for which

$$u(\tau, \epsilon) = O(\epsilon^{N+1}),\ v(\tau, \epsilon) = O(\epsilon^{N+1}),\ w(\tau, \epsilon) = O(\epsilon^{N+1}), \tag{7.34}$$

uniformly for $0 \leq \tau \leq T/\epsilon$.

With the change of variables

$$\begin{aligned}
\hat{u}(t, \epsilon) &= u(t/\epsilon, \epsilon), \\
\hat{v}(t, \epsilon) &= v(t/\epsilon, \epsilon), \\
\hat{w}(t, \epsilon) &= w(t/\epsilon, \epsilon), \\
\tau &= t/\epsilon, \quad s = s'/\epsilon,
\end{aligned}$$

the constrained integral equation (7.29) becomes

$$\begin{aligned}
\hat{u}(t, \epsilon) &= \hat{\Phi}(t, 0, \epsilon)\tilde{\theta}_1(\epsilon) + \int_0^t \hat{\Phi}(t, s', \epsilon)(L_1(s'/\epsilon)\hat{v}(s', \epsilon) + E_1(s'/\epsilon)\hat{w}(s', \epsilon) \\
&\qquad + \hat{G}_1(\epsilon, \hat{u}(s', \epsilon), \hat{v}(s', \epsilon), \hat{w}(s', \epsilon), s'))ds' \\
\hat{v}(t, \epsilon) &= \hat{\Psi}(t, \epsilon)\tilde{\theta}_2(\epsilon) + \int_0^t \frac{\hat{\Psi}(t, \epsilon)\hat{\Psi}^{-1}(s', \epsilon)}{\epsilon}(E_2(s'/\epsilon)\hat{w}(s', \epsilon) \\
&\qquad + \hat{G}_2(\epsilon, \hat{u}(s', \epsilon), \hat{v}(s', \epsilon), \hat{w}(s', \epsilon), s'))ds' \\
\hat{u}(t, \epsilon) &= -E_3^{-1}(t/\epsilon)\hat{G}_3(\epsilon, \hat{u}(t, \epsilon), \hat{v}(t, \epsilon), \hat{w}(t, \epsilon), t)
\end{aligned} \tag{7.35}$$

where

$$\begin{aligned}
&\hat{\Phi}(t, s', \epsilon) = \Phi(t/\epsilon, s/\epsilon, \epsilon) \\
&\hat{\Psi}(t, \epsilon) = \Psi(t/\epsilon) \\
&\hat{G}_i(\epsilon, \hat{u}, \hat{v}, \hat{w}, t) = G_i(\epsilon, u, v, w, t/\epsilon), \qquad \text{for } i = 1, 2, 3.
\end{aligned}$$

Because of (7.33) the kernel in the second equation in (7.35) satisfies

$$\|\hat{\Psi}(t,\epsilon)\hat{\Psi}^{-1}(s',\epsilon)\| \leq Ke^{-\frac{\mu(t-s')}{2\epsilon}} \tag{7.36}$$

for all $0 \leq s' \leq t \leq T$. Moreover $\hat{\Phi}(t,s',\epsilon)$, $L_1(s'/\epsilon)$, $E_i(s'/\epsilon)$, $i = 1,2,3$, are uniformly bounded in $0 \leq s' \leq t \leq T$. Since G_i, $i = 1,2,3$, satisfy (7.14a,b), we see that $\hat{G}_i(\epsilon,\hat{u},\hat{v},\hat{w},t)$, $i = 1,2,3$, satisfy the **Condition (N)**. Now, with the same notation as in (5.15) we rewrite (7.25) in the form of constrained systems of integral equations as the one in [Ya2]. Then Theorem 1,2 in [Ya2] guarantee that the system (7.35) has a unique solution $(u(\hat{t},\epsilon), v(\hat{t},\epsilon), w(\hat{t},\epsilon))$ on the interval $0 \leq t \leq T$ for which

$$\left.\begin{aligned} \hat{u}(t,\epsilon) &= O(\epsilon^{N+1}) \\ \hat{v}(t,\epsilon) &= O(\epsilon^{N+1}) \\ \hat{w}(t,\epsilon) &= O(\epsilon^{N+1}) \end{aligned}\right\} \quad \text{as } \epsilon \to 0,$$

uniformly on $0 \leq t \leq T$. Hence the system (7.29) has a unique solution $(u(\tau,\epsilon), v(\tau,\epsilon), w(\tau,\epsilon))$ on the interval $0 \leq \tau \leq T/\epsilon$ which satisfies the estimate (7.34). This completes the proof of Theorem 7.1. ∎

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In this paper, singularly perturbed nonlinear differential/algebraic equations (DAEs) are considered and a proof of the existence and uniqueness of a solution is given. Asymptotic expansions for such a solution are obtained and proved to be uniformly convergent. This generalizes known results about asymptotic expansions of singularly perturbed ordinary differential equations.

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